Probabilistic Error Analysis of Limited Precision Stochastic Rounding

joint work with El-Mehdi El Arar, Mantas Mikaitis, and Massimiliano Fasi

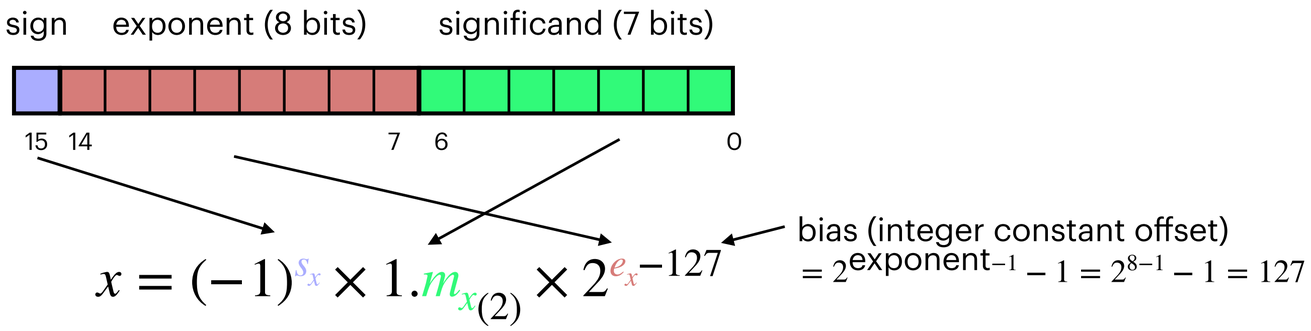
Silviu-Ioan Filip, 1 September 2025

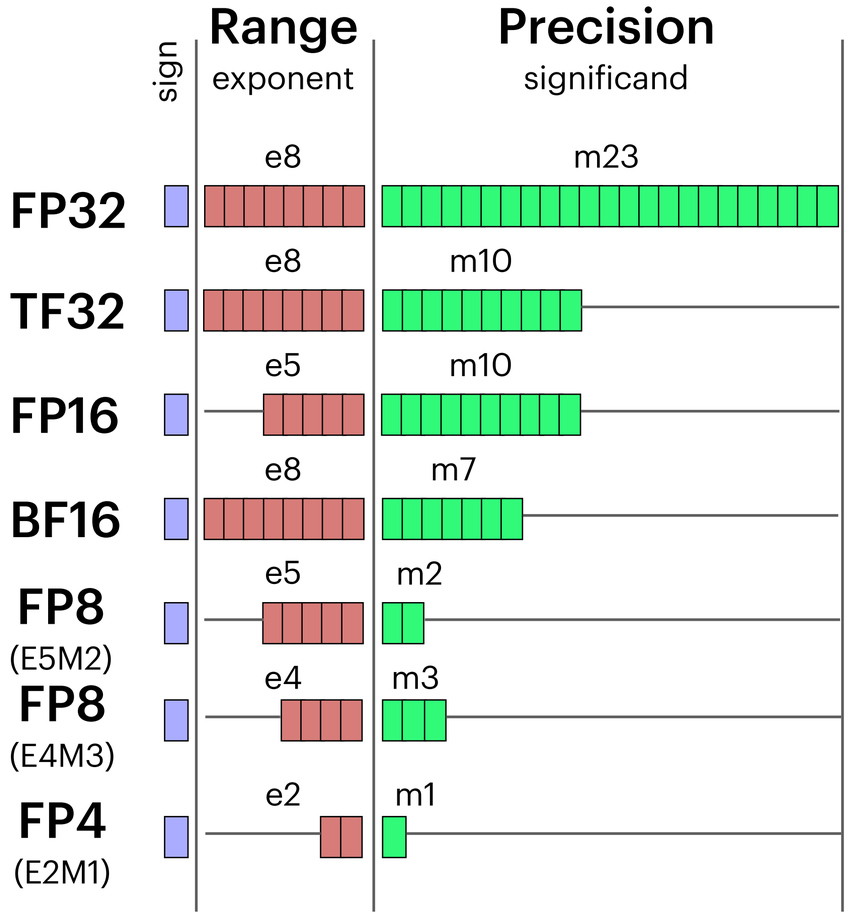
Common and Emerging FP Formats in HPC & AI

➡they offer various tradeoffs in terms of range, precision & performance

FP performance numbers for recent NVIDIA GPU architectures

Peak performance (TFLOPS)							
Device	Year	fp64	fp32	tfloat32	fp16	bfloat16	fp8
P100	2016	5	9	-	19	-	-
V100	2017-2019	8	16	-	125	-	-
A100	2020-2021	19	19	156	312	312	-
H100	2022	48	48	400	800	800	1600





IEEE FP Standardisation for ML: P3109 WG

https://github.com/P3109/Public

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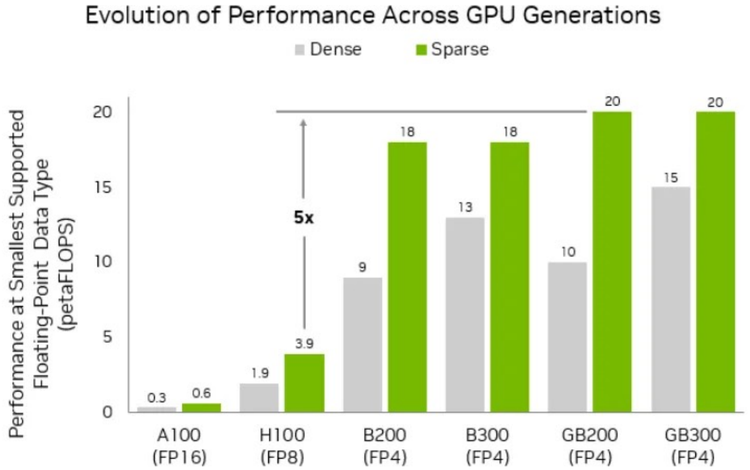

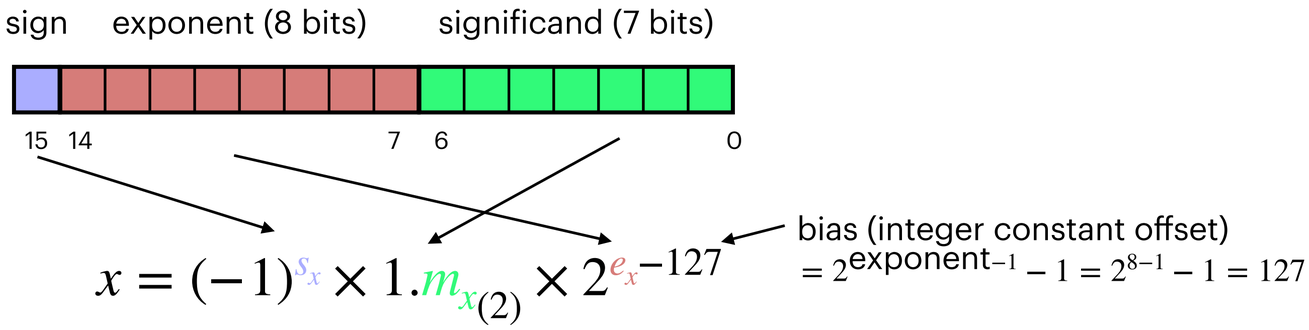

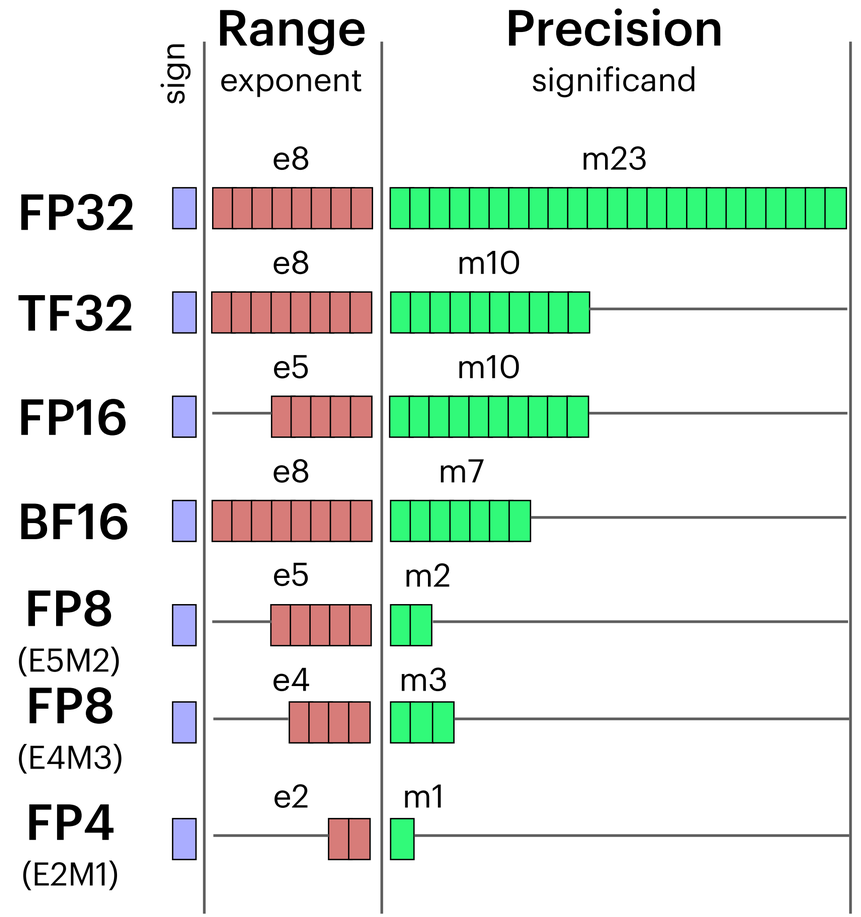

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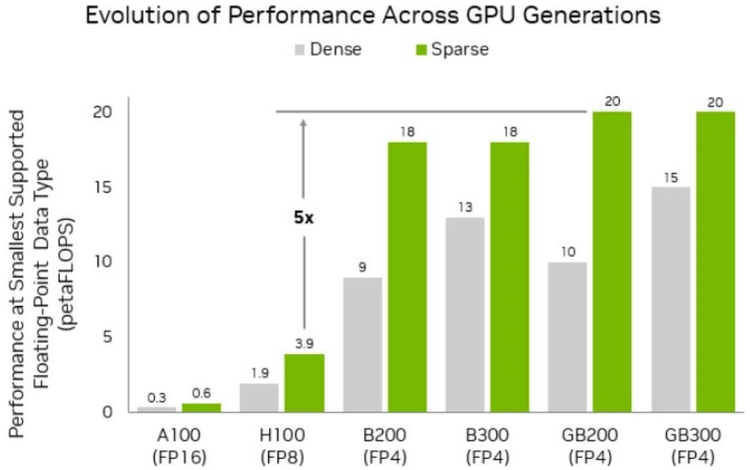
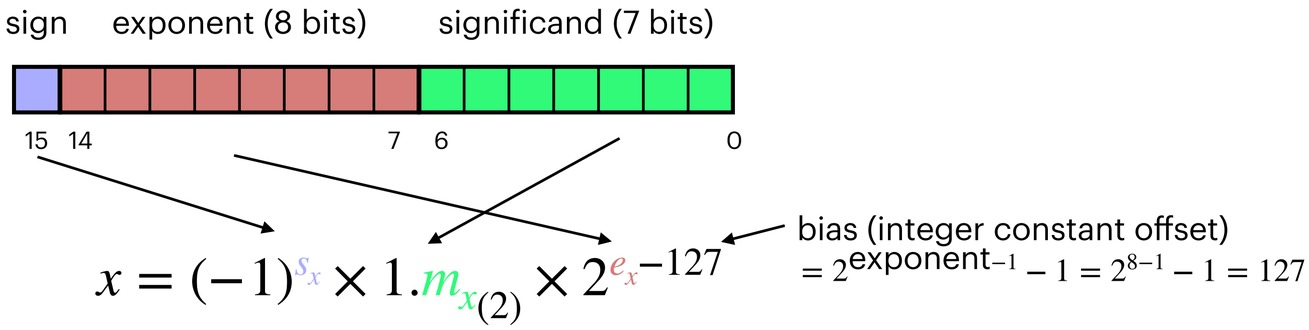
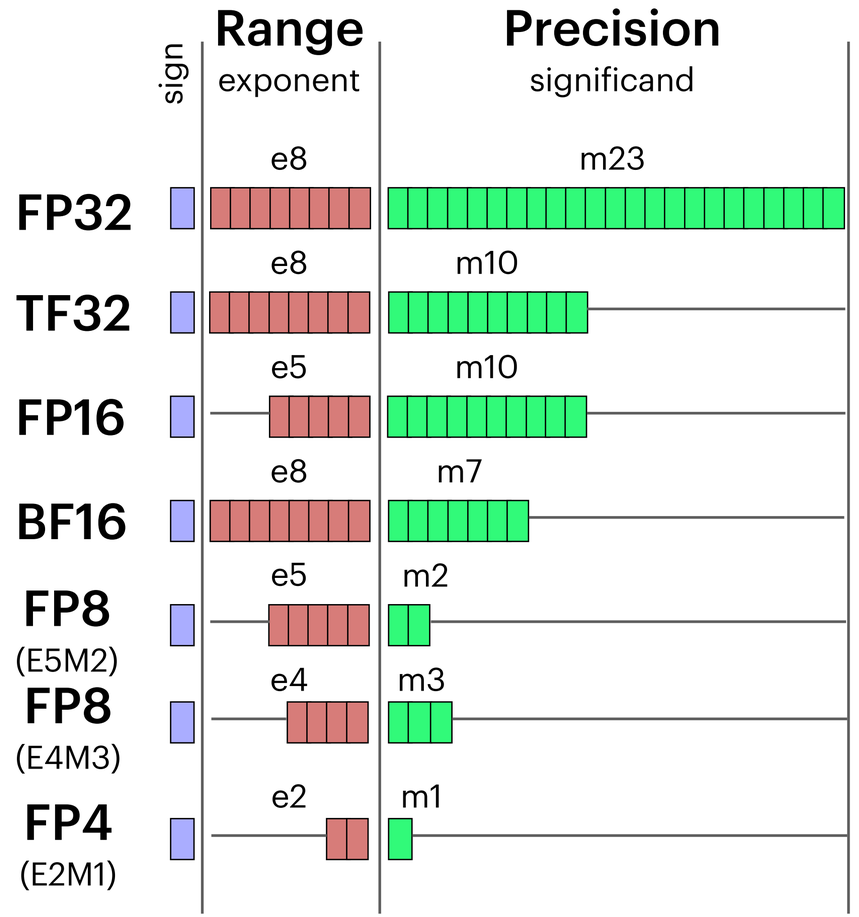




➡effective rounding modes are critical: **stochastic rounding**



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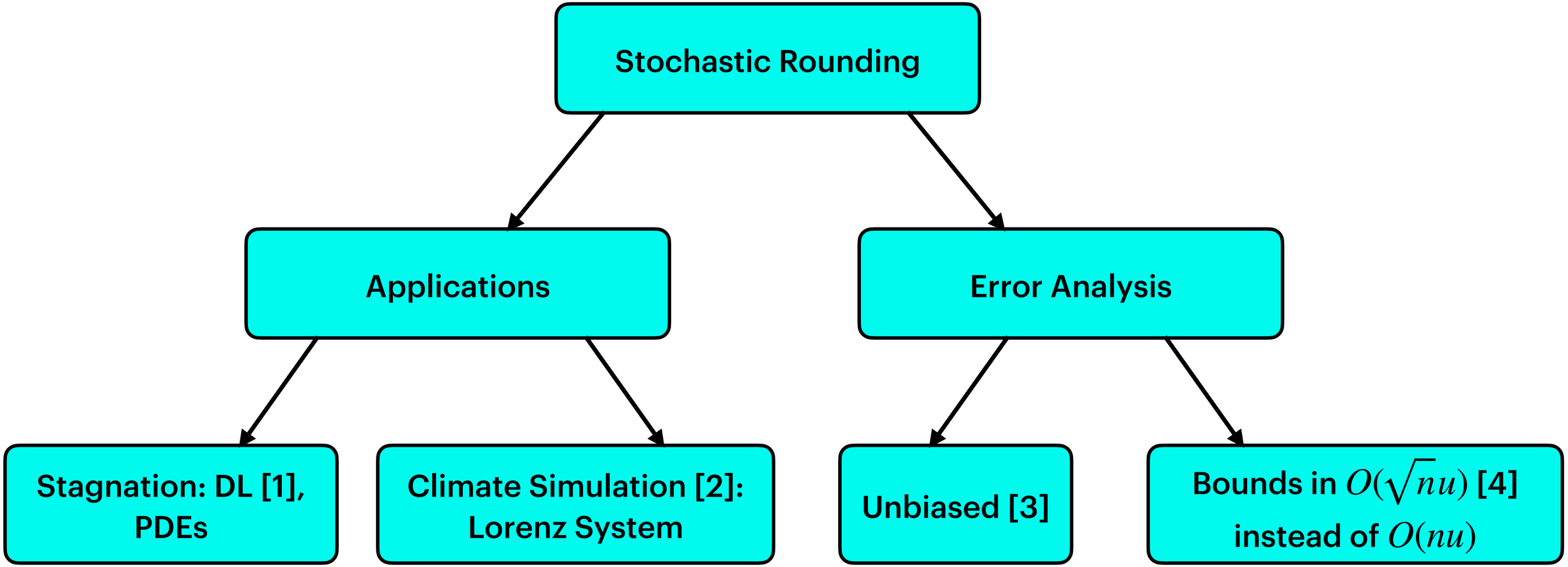
Motivation



[1] Gupta et al: Deep Learning with Limited Numerical Precision
[2] Paxton et al: Climate Modeling in Low Precision: Effects of Both Deterministic and Stochastic Rounding
[3] Parker: Monte Carlo Arithmetic: Exploiting Randomness in Floating-Point Arithmetic
[4] El Arar: Stochastic Models for the Evaluation of Numerical Errors

Stochastic Rounding

For $x, y \in \mathbb{R}$ and $\text{op} \in \left\{+, -, \times, /, \sqrt{}\right\}$

$$\text{SR}_p(x) = x(1+\delta), \qquad |\delta| \leqslant u_p := 2^{1-p}$$

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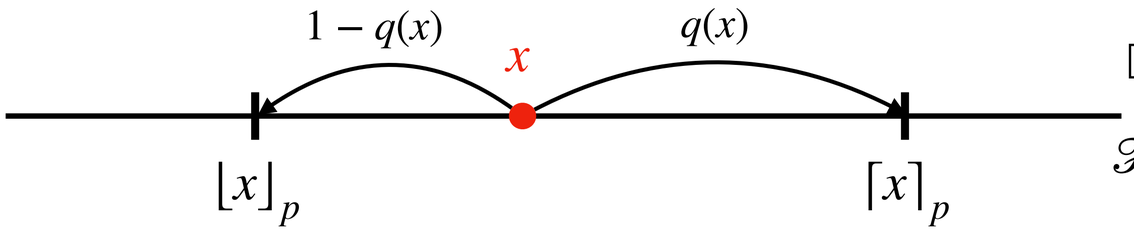

$\mathbb{R}$

$\mathscr{F}_p$

$$\text{SR}_p(x) = \begin{cases} \lceil x \rceil_p, & \text{with prob. } q(x) \\ \lfloor x \rfloor_p, & \text{with prob. } 1 - q(x) \end{cases} \qquad q(x) = \frac{x - \lfloor x \rfloor_p}{\lceil x \rceil_p - \lfloor x \rfloor_p}$$

- Introduced by Forsythe in 1950

not exceeding u. The present author proposes a *random round-off*, whereby any real number $u = [u] + v$ is "rounded up" to $[u]+1$ with probability v, and "rounded down" to $[u]$ with probability $1-v$. The error r of random round-off is truly a random variable. Since $E(r) = 0$ and $E(r^2) = v(1-v) \leqq 1/2$, simple probabilistic bounds on the accumulated error can be given without assuming a distribution for v. Tests with

- Of interest in recent years due to use in Deep Learning

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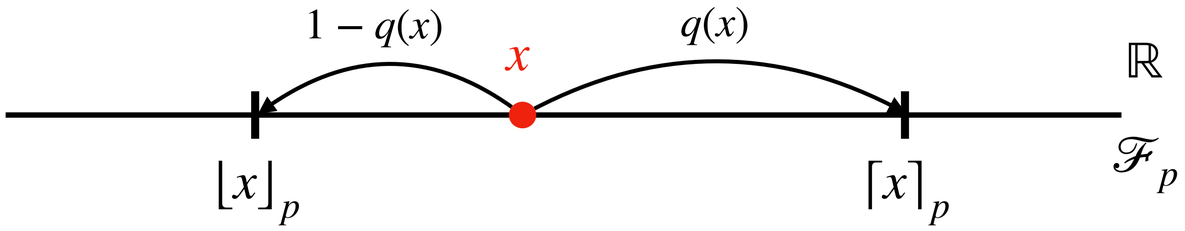

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Emerging HW support

- Graphcore IPUs
- AMD MI300 GPUs
- Tesla D1
- Amazon Trainium
- NVIDIA Blackwell

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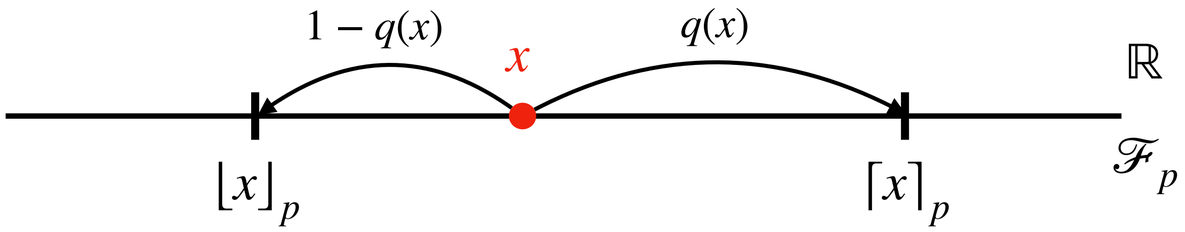

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Advantages

+ roundoff errors with zero mean & error cancellation in long comp. chains
+ tends to avoid stagnation effects
+ total roundoff error grows more slowly with problem size n than with classic deterministic rounding (e.g. round to nearest (RN))

$$O\left(\sqrt{n}u_p\right) \text{vs } O\left(nu_p\right)$$

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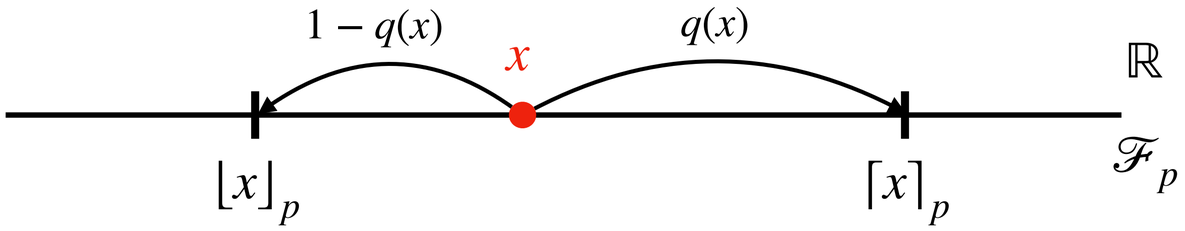

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Limitations

– lack of reproducibility

– increased computational overhead

– inability to use true randomness

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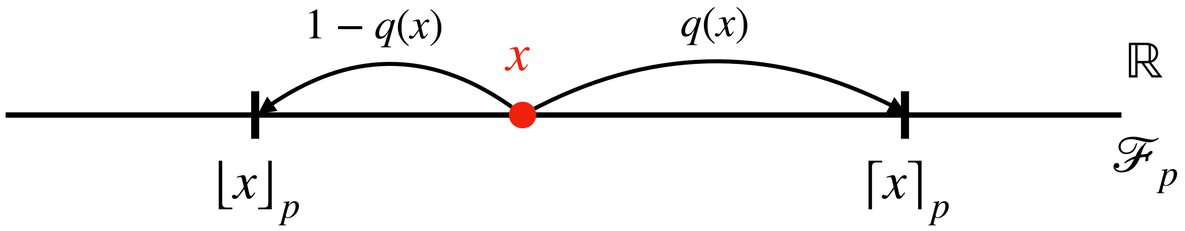

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$O\left(\sqrt{n}u_p\right)$ vs $O\left(nu_p\right)$

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How can we show this?

Ingredients

Definition (*Mean Independence*) A rv X is ***mean independent*** of the rv Y if $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$. The sequence of rvs $X_0, X_1, \ldots$ is ***mean independent*** if $\mathbb{E}(X_k \mid X_0, \ldots, X_{k-1}) = \mathbb{E}(X_k)$ for all k.

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Definition (*Martingale*) A sequence of rvs $M_0, \ldots, M_n$ is a ***martingale*** with respect to the sequence of rvs $X_0, \ldots, X_n$ if, for all k:

(a) M_k is a function of $X_0, \ldots, X_k$;

(b) $\mathbb{E}(|M_k|) < \infty$;

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Lemma (*Azuma-Hoeffding*) Let $M_0, \ldots, M_n$ be a martingale wrt $X_0, \ldots, X_n$. We assume there exist $a_k > 0$ s.t. $|M_k - M_{k-1}| \leqslant a_k$, for $k = 1, \ldots, n$. Then, for any $\lambda > 0$

$$\mathbb{P}\left(|M_n - M_0| \geqslant \lambda \sqrt{\sum_{k=1}^{n} a_k^2} \right) \leqslant 2 \exp\left(-\lambda^2/2\right).$$

Proof Sketch [1]

Mean independence of rounding errors

The computations of interest generate rounding errors $\delta_1, \delta_2, \ldots$

The δ_k are rvs of mean zero such that

$$\mathbb{E}(\delta_k \,|\, \delta_1, \ldots, \delta_{k-1}) = \mathbb{E}(\delta_k) = 0$$

[1] Stochastic rounding and its probabilistic backward error analysis, M. P. Connolly, N. J. Higham, T. Mary, SISC, vol. 43, No. 1, pp. A565-A585, 2021

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Proof Sketch [1]

Mean independence of rounding errors

Obtain a **martingale**

Get $O\left(\sqrt{n}u_p\right)$ probabilistic error bound using **Azuma-Hoeffding**

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$$\prod_{i=1}^{n}(1+\delta_i) = 1 + \theta_n, |\,\theta_n\,| \leqslant \lambda\sqrt{n}u_p + O(u_p^2)$$ with probability at least $1 - 2\exp\left(-\lambda^2/2\right)$

If $y = a^T b$, where $a, b \in \mathbb{R}^n$, is evaluated using SR_p, then the computed result $\hat{y}$ satisfies

$$\frac{|\,y - \hat{y}\,|}{|\,y\,|} \leqslant \frac{|\,a\,|^T |\,b\,|}{|\,a^T b\,|}\left(\lambda\sqrt{n}u_p + O(u_p^2)\right)$$

with the same probability

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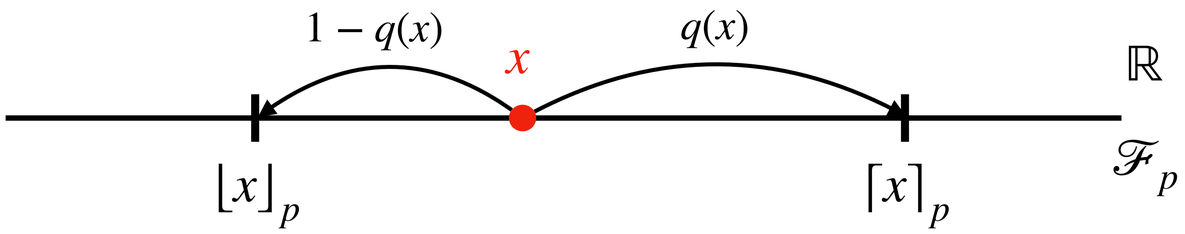

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What do we need to implement it?

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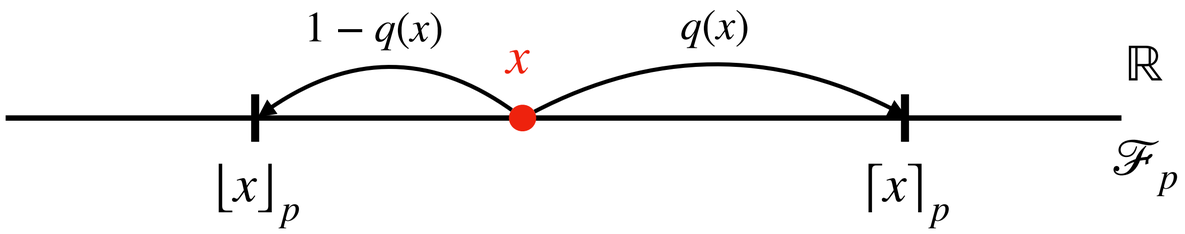

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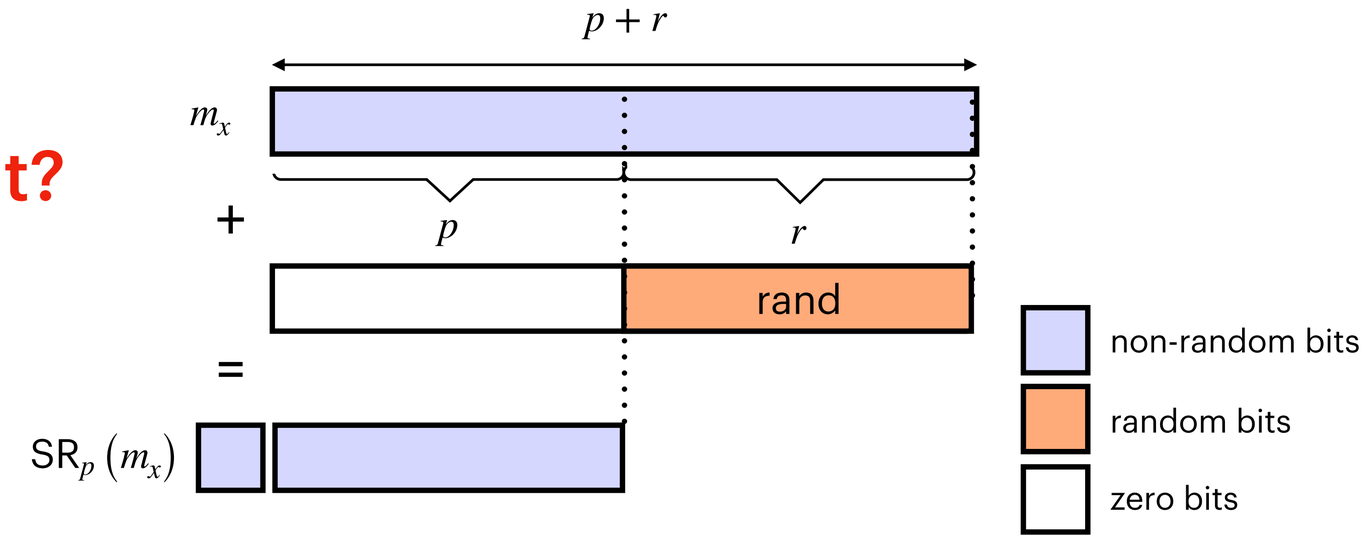
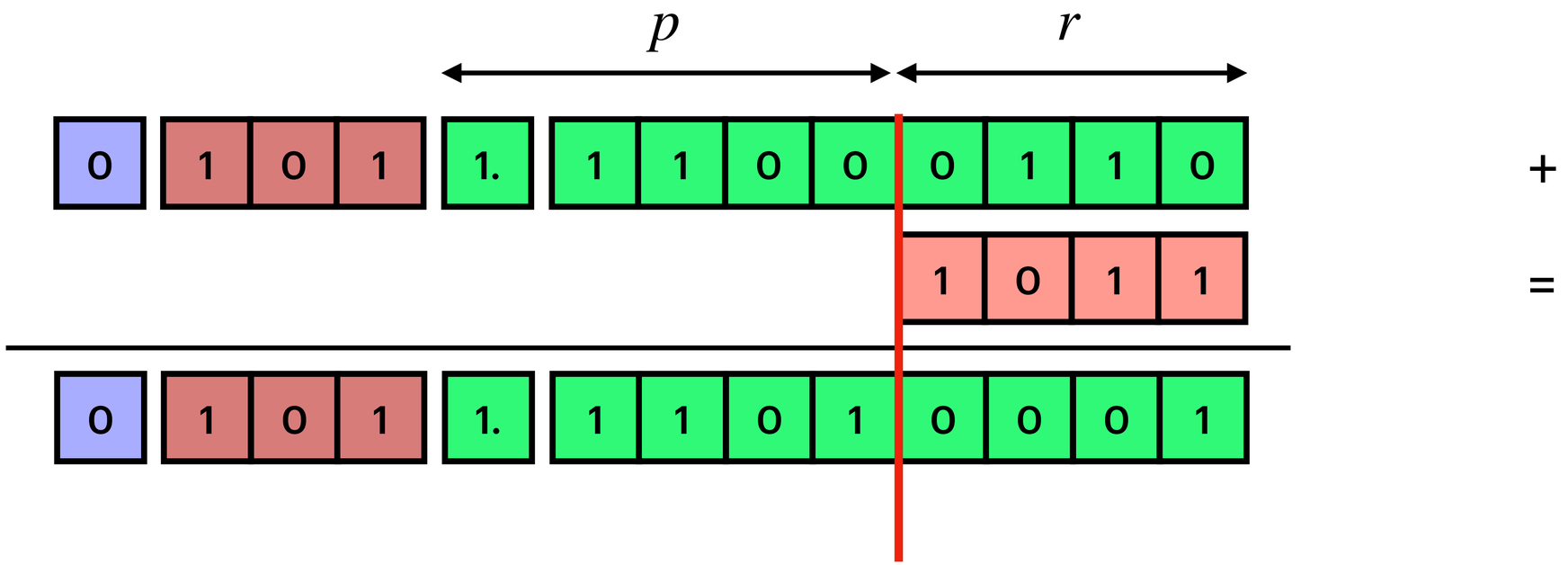
What do we need to implement it?

- A stream of random bits
- An accurate view of x (or x op y)



SR Example: E3M4 ($p = 5$)



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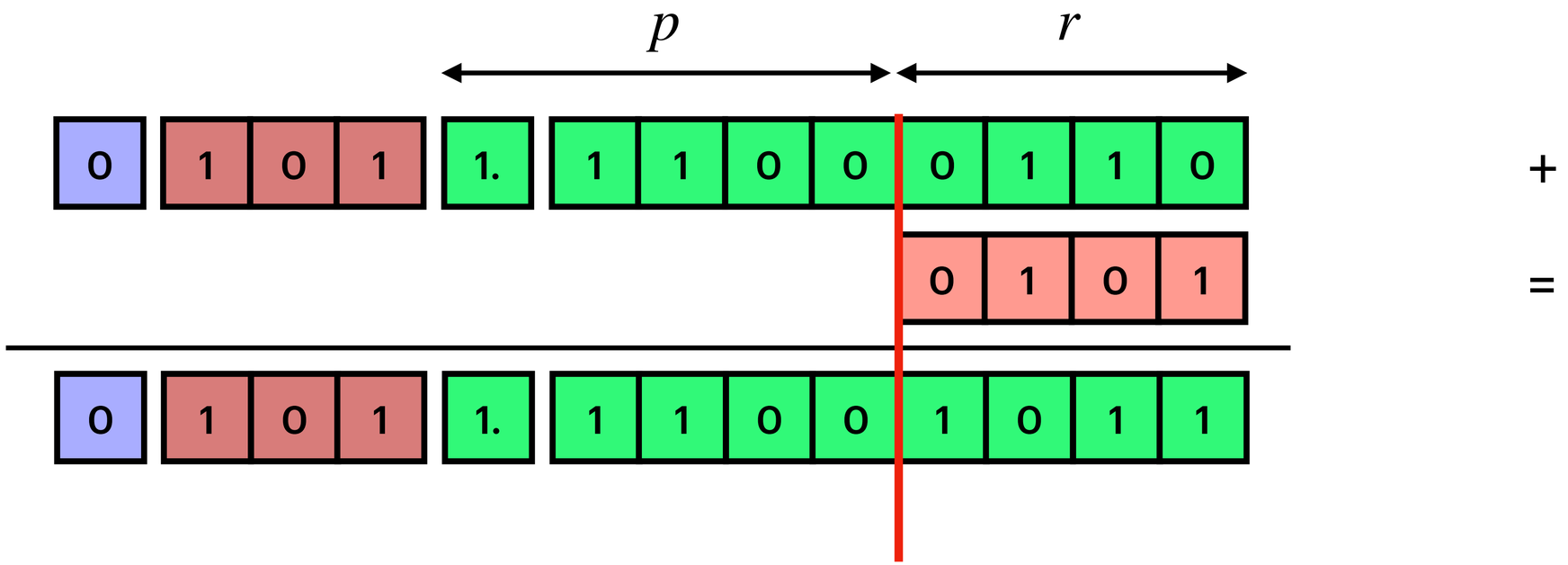

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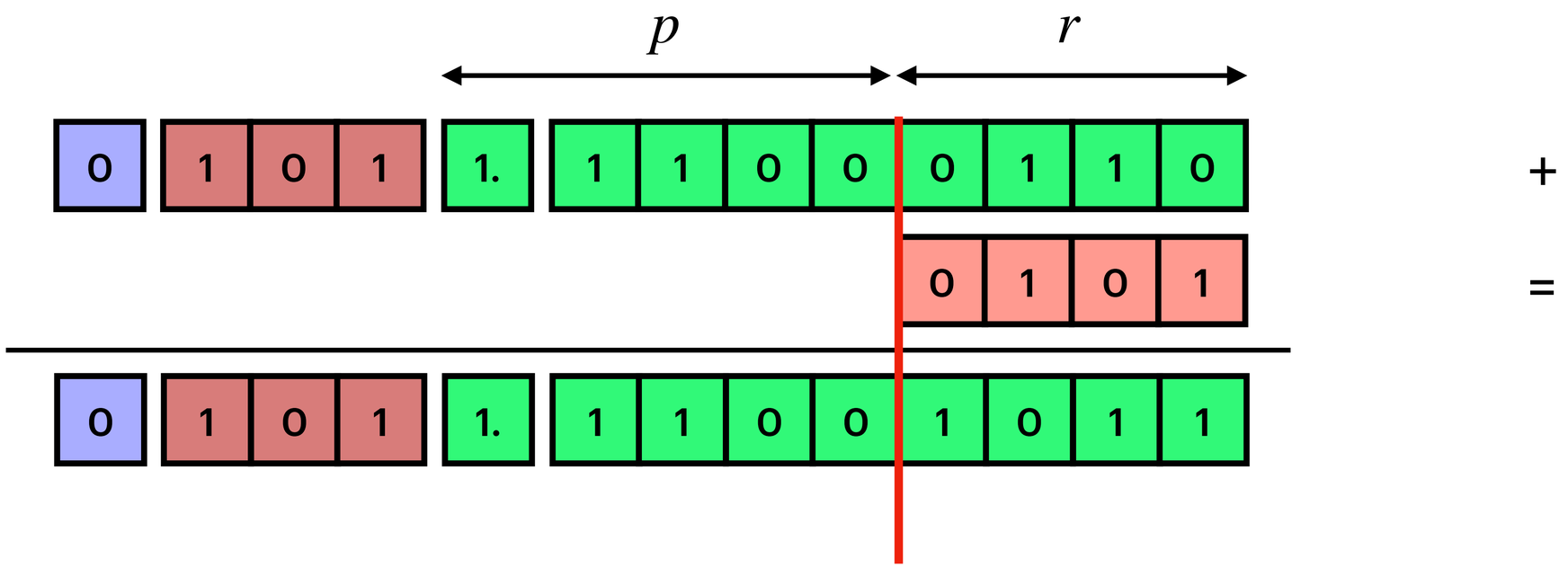
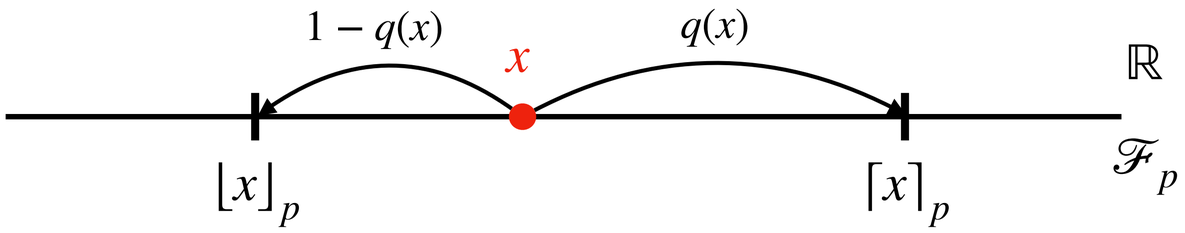


How many random bits r (i.e., extra precision) are needed?

Limited Precision Stochastic Rounding



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Exact

$\mathbb{E}(\mathrm{SR}_p(x)) = x$ (unbiased)

$\mathbb{V}(\mathrm{SR}_p(x)) \leqslant x^2 u_p^2 / 4$

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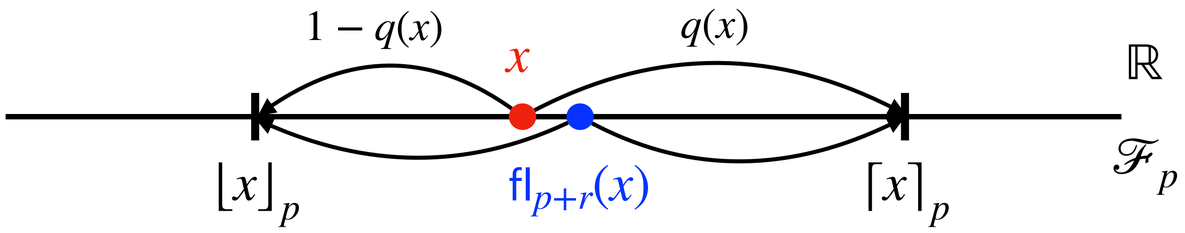

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Limited Precision

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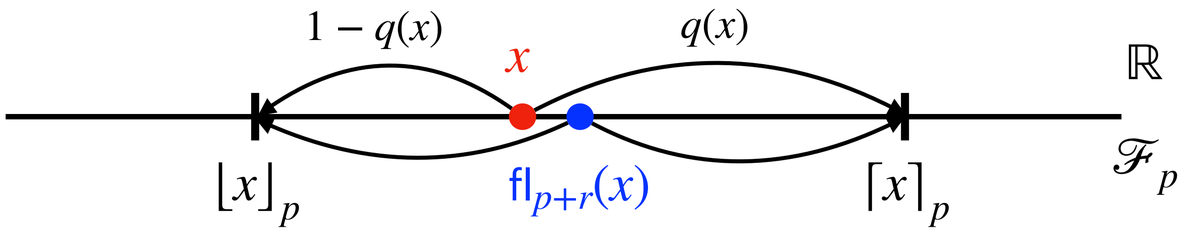

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- $\mathrm{fl}_{p+r}(x) = x(1+\beta)$, such that $|\beta| \leqslant u_{p+r}$
- $\mathrm{SR}_{p,r}(x) = x(1+\delta)$, such that $|\delta| \leqslant u_p$
- mean independence property is lost

$$\mathbb{E}(\delta_k \,|\, \delta_1, \ldots, \delta_{k-1}) = \beta_k \neq \mathbb{E}(\delta_k)$$

- β_k is a rv and $\mathbb{E}(\beta_k) = \mathbb{E}(\delta_k)$

Sufficient values for r

• Conversion of precision $t > p$ number	$t - p$
• Addition	$e_{\max} - e_{\min}$
• Multiplication	p
• Division	unbounded

How do smaller r affect the $O(\sqrt{n}u_p)$ bound?

Limited Precision Stochastic Rounding

We can show the following result about mean independence

Lemma [1]. Let $\delta_1, \ldots, \delta_n$ be random errors produced by a sequence of elementary ops using $\mathrm{SR}_{p,r}$ and let $\beta_1, \ldots, \beta_n$ be their corresponding errors incurred by fl_{p+r}.
Then, the rvs $\alpha_k = \delta_k - \beta_k$, for $1 \leqslant k \leqslant n$ are **mean independent**

$$\mathbb{E}(\alpha_k \,|\, \alpha_1, \ldots, \alpha_{k-1}) = \mathbb{E}(\alpha_k) = 0$$

Moreover, for all $1 \leqslant i \leqslant n$,

$$\prod_{k=i}^{n}(1+\delta_k) = \prod_{k=i}^{n}(1+\alpha_k) + \mathscr{B}_i,$$

with $|\mathscr{B}_i| \leqslant \gamma_{n-i+1}(u_p + u_{p+r}) - \gamma_{n-i+1}(u_p)$, where $\gamma_m(x) = (1+x)^m - 1$

[1] Probabilistic error analysis of limited-precision stochastic rounding, E.-M. El Arar, M. Fasi, S.-I. Filip, M. Mikaitis, arXiv 2408.03069, to appear in SISC

Limited Precision Stochastic Rounding

Theorem [1]. If $y = a^T b$, where $a, b \in \mathbb{R}^n$, is evaluated using $\mathrm{SR}_{p,r}$, then the computed result $\widehat{y}$ satisfies

$$\frac{|y - \widehat{y}|}{|y|} \leqslant \frac{|a|^T |b|}{|a^T b|} \left(\lambda \sqrt{n} u_p + n u_{p+r} + O(u_p^2 + u_{p+r}^2) \right)$$

with probability at least $1 - 2\exp(-\lambda^2/2)$.

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Proof Idea.

$$\prod_{k=i}^{n} (1 + \delta_k) = \prod_{k=i}^{n} (1 + \alpha_k) + \mathscr{B}_i,$$

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Proof Idea.

$$\prod_{k=i}^{n} (1 + \delta_k) = \prod_{k=i}^{n} (1 + \alpha_k) + \mathscr{B}_i,$$

Martingale argument

$$O(\sqrt{n} u_p)$$

$$|\mathscr{B}_i| \leqslant \gamma_{n-i+1}(u_p + u_{p+r}) - \gamma_{n-i+1}(u_p)$$

$$O(n u_{p+r})$$

$$O(\sqrt{n} u_p + n u_{p+r})$$

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with probability at least $1 - 2\exp(-\lambda^2/2)$.

$$O(\sqrt{n} u_p + n u_{p+r})$$

Good rule of thumb for choosing r in length n summation chains:

$$\sqrt{n} u_p \geqslant n u_{p+r} \Rightarrow r \geqslant \lceil (\log_2 n)/2 \rceil$$

[1] Probabilistic error analysis of limited-precision stochastic rounding, E.-M. El Arar, M. Fasi, S.-I. Filip, M. Mikaitis, arXiv 2408.03069, to appear in SISC

Numerical Experiments

- Recursive summation (srfloat)
- Rosenbrock function minimisation (srfloat)
- Parameter update in DNN training (mptorch)

SR$_{p,r}$ simulation tools:

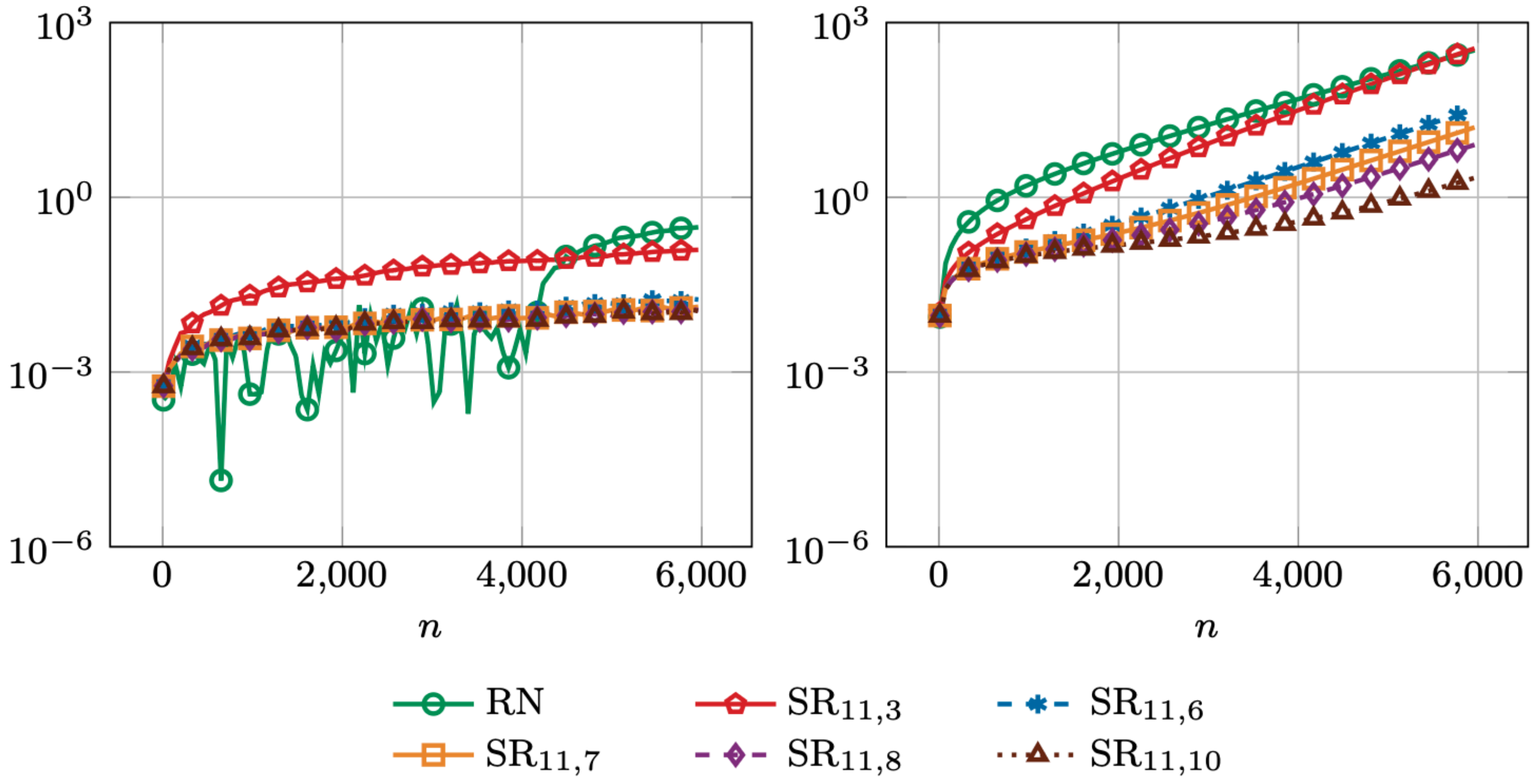
- https://github.com/sfilip/srfloat
- https://github.com/mptorch/mptorch

Recursive Summation

- binary16 ($p = 11$) arithmetic
- elements sampled from $U([0,1])$
- r close to $\lceil (\log_2 6{,}000)/2 \rceil = 7$ works well

Relative forward error

Bounds with $1 - 2\exp(-\lambda^2/2) = 0.9$



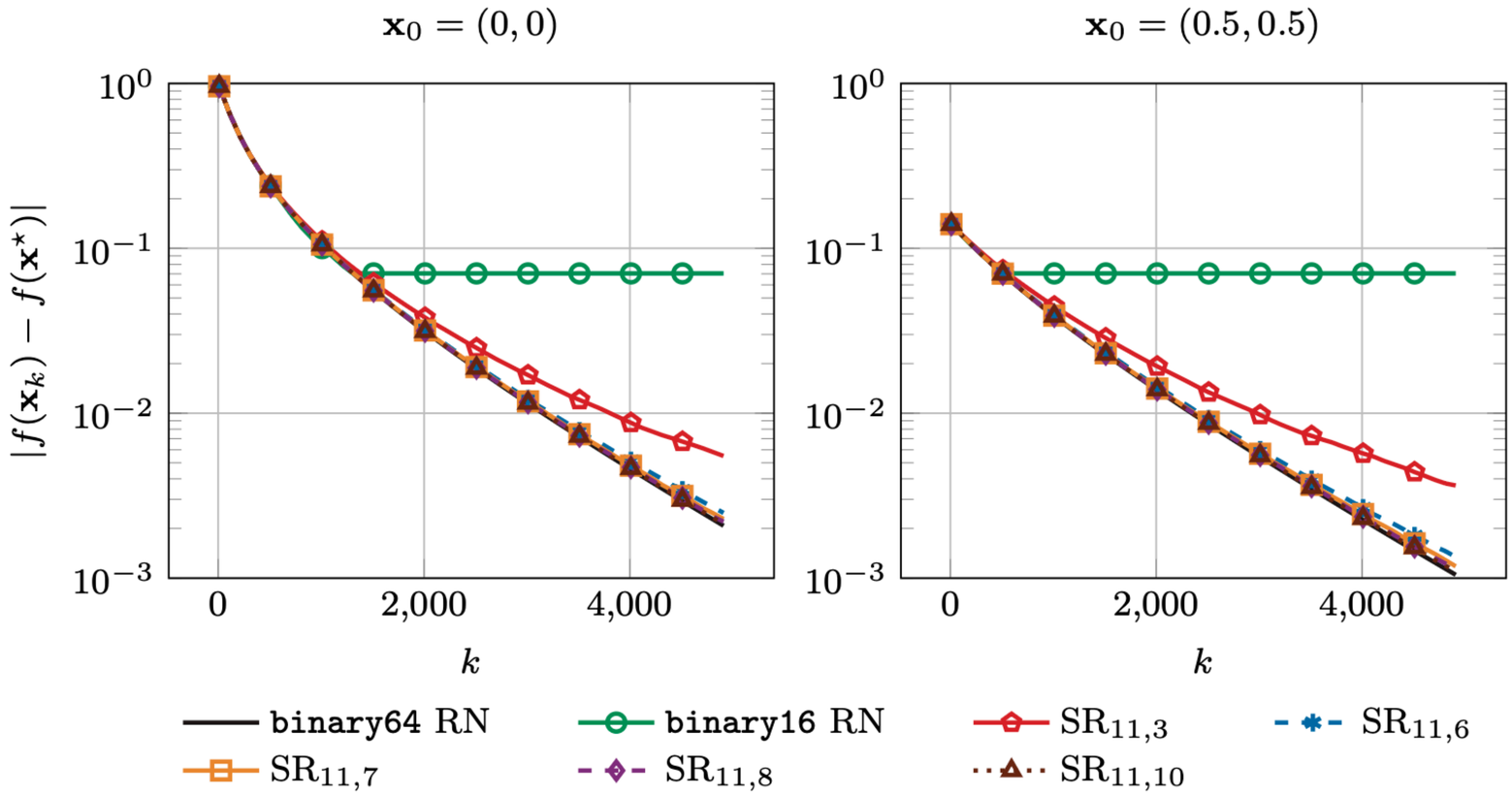
Rosenbrock Function

$$f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2,$$

- GD update rule ($t_k = 0.001$):

$$\mathbf{x}_{k+1} = \mathbf{x}_k - t_k \nabla f(\mathbf{x}_k),$$

- r close to $\lceil (\log_2 6{,}000)/2 \rceil = 7$ works well



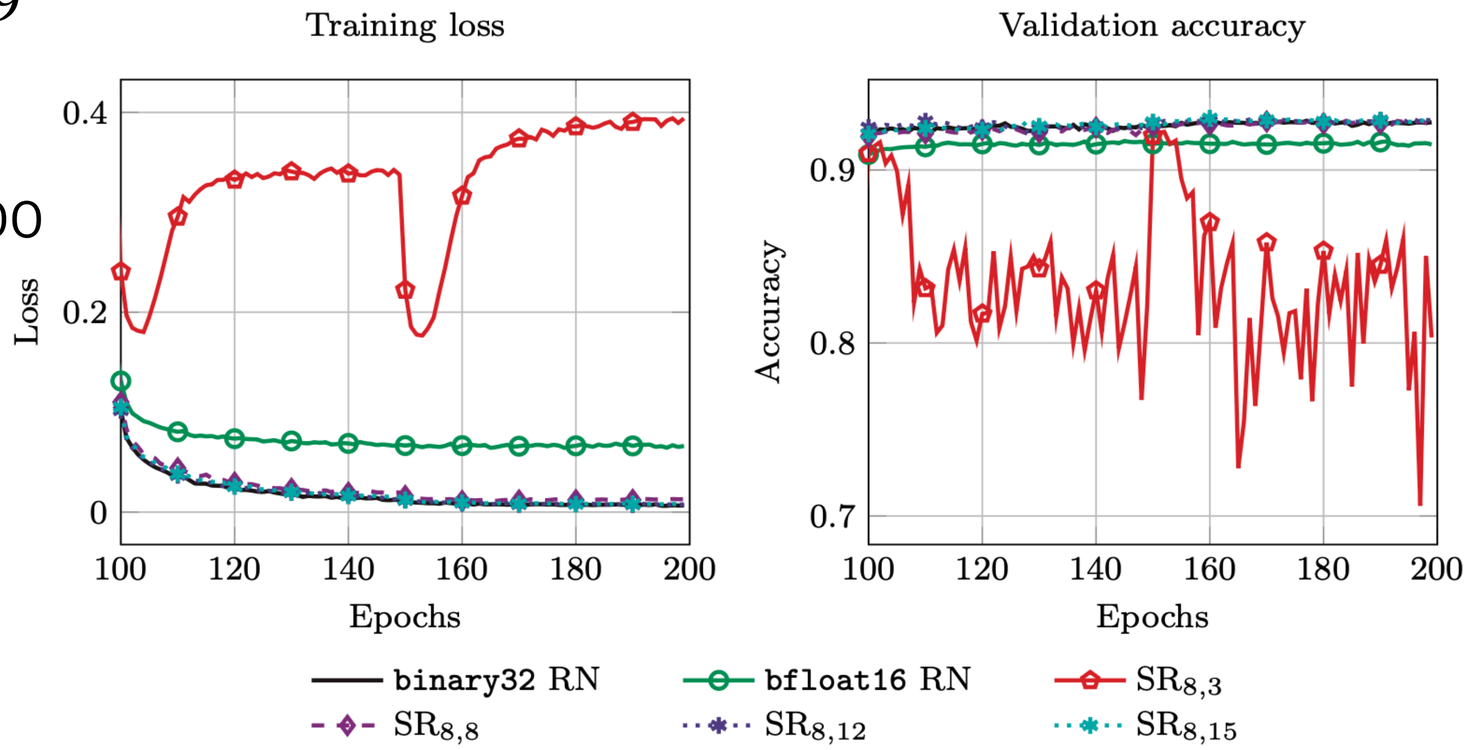
Parameter Updates in DNN Training

- Training a ResNet32 model on CIFAR-10
- Hyperparameters
  - Batch size: 128, momentum: $\mu = 0.9$
  - Training time: 64,000 iterations (200 epochs)
  - LR: $t_k = 0.1$, reduced by 10 at 32,000 and 48,000 iterations

- bfloat16 ($p = 8$) arithmetic
- Update rule:

$$\mathbf{v}_{k+1} = \circ(\mu \mathbf{v}_k + \mathbf{g}_k),$$
$$\mathbf{x}_{k+1} = \circ(\mathbf{x}_k - t_k \mathbf{v}_{k+1}),$$

$$r \geqslant \left\lceil (\log_2 64{,}000)/2 \right\rceil = 8$$



MPTorch: Simulation Support for MP Training

➡library/simulation support for low/mixed precision training is limited

- **QPyTorch** [1]: applies quantization on operands for FP32 arithmetic kernels

Limitation: not an adequate simulation of low precision training hardware

[1] QPyTorch: A Low-Precision Arithmetic Simulation Framework, *Zhang et al.,* arXiv:1910.04540, 2019
[2] MPTorch and MPArchimedes: Open Source Frameworks to Explore Custom Mixed-Precision Operations for DNN Training on Edge Devices, Tatsumi et al., ROAD4NN 2021
[3] MPTorch-FPGA: a Custom Mixed-Precision Framework for FPGA-based DNN Training,*S. Ben Ali, S.-I. Filip, O. Sentieys. G. Lemieux*, IEEE/ACM DATE conference, Mar 2025

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MPTorch [2,3]

➡PyTorch-based extension for low/mixed precision training (& inference) simulation

- mix of C++, CUDA C++, HLS, and Python
- **precision and format at the operator level** for **inference** (FWD) and **training** (BWD)
- facilitate research into low/mixed precision DNN training accelerators

➡ support for various number formats and rounding modes:

- floating-point, fixed-point, block floating-point, logarithmic,...
- round to nearest, stochastic rounding,...

➡ also: Transformer-based architectures + IEEE-P3109 WG FP8 formats

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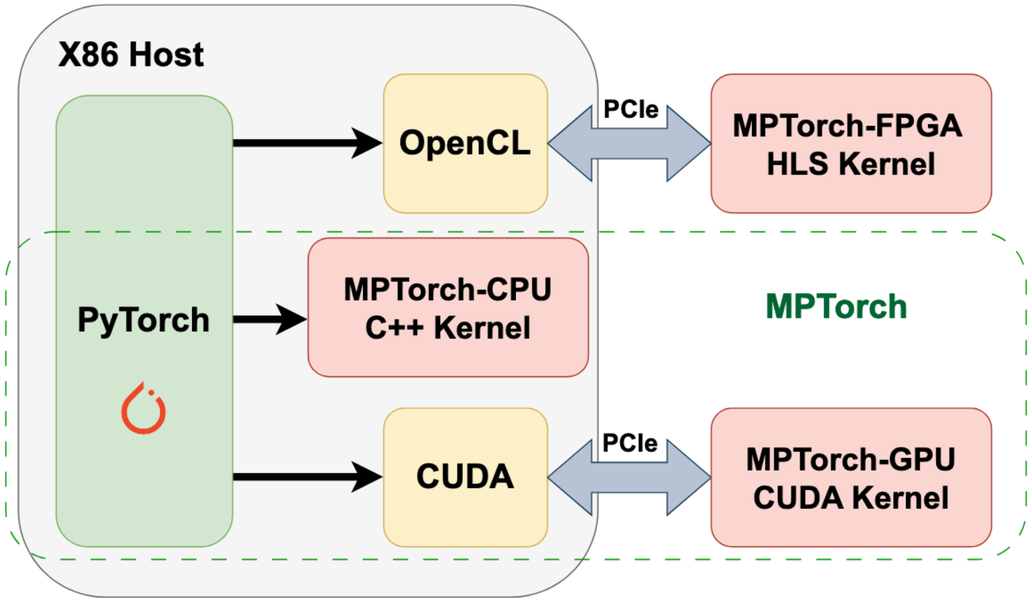

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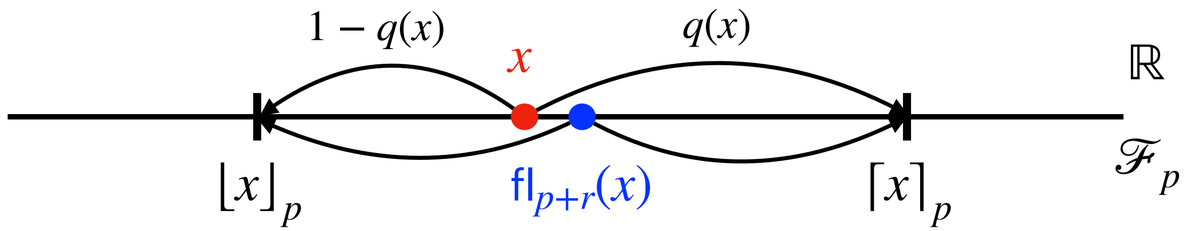
Limited Precision Stochastic Rounding [1]



$$\mathrm{SR}_p(x) = \begin{cases} \lceil x \rceil_p, & \text{with prob. } q(x) \\ \lfloor x \rfloor_p, & \text{with prob. } 1-q(x) \end{cases} \qquad q(x) = \frac{x - \lfloor x \rfloor_p}{\lceil x \rceil_p - \lfloor x \rfloor_p}$$

$$\mathrm{SR}_{p,r}(x) = \begin{cases} \lceil x \rceil_p, & \text{with prob. } q_r(x) \\ \lfloor x \rfloor_p, & \text{with prob. } 1-q_r(x) \end{cases} \qquad q_r(x) = \frac{\mathsf{fl}_{p+r}(x) - \lfloor x \rfloor_p}{\lceil x \rceil_p - \lfloor x \rfloor_p}$$

Exact

$\mathbb{E}(\mathrm{SR}_p(x)) = x$ (unbiased)

$\mathbb{V}(\mathrm{SR}_p(x)) \leqslant x^2 u_p^2/4$

$O(\sqrt{n}u_p)$ bounds

Limited Precision

$\mathbb{E}(\mathrm{SR}_{p,r}(x)) = \mathsf{fl}_{p+r}(x)$ (biased)

$\mathbb{V}(\mathrm{SR}_{p,r}(x)) \leqslant x^2 u_p^2/4$

$O(\sqrt{n}u_p + nu_{p+r})$ bounds

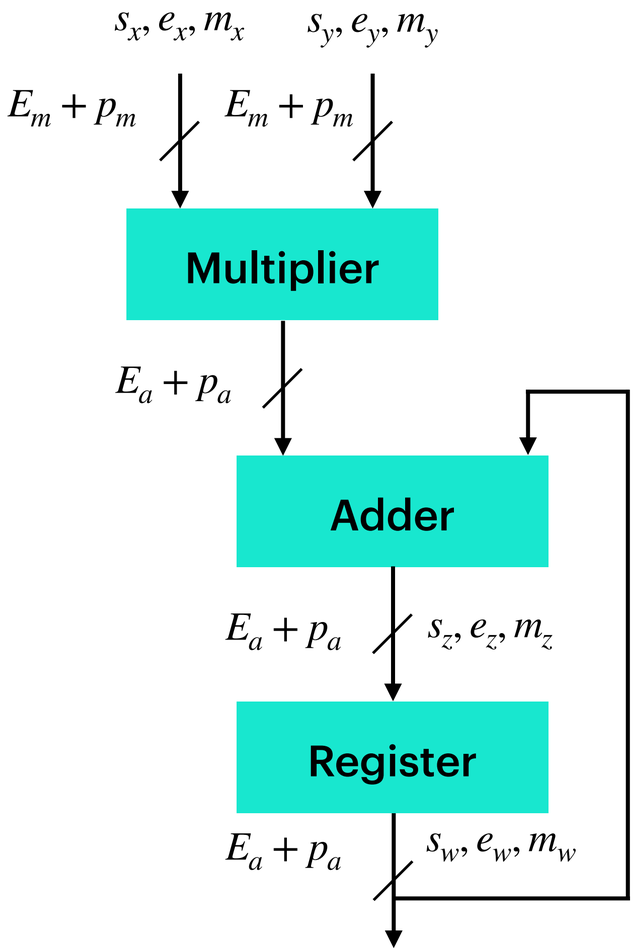
$r \geqslant \lceil (\log_2 n)/2 \rceil$

Thank you! Questions?

[1] Probabilistic error analysis of limited-precision stochastic rounding, E.-M. El Arar, M. Fasi, S.-I. Filip, M. Mikaitis, arXiv 2408.03069

Classic MAC Unit Design

Multiply-Accumulate (MAC) Unit Design

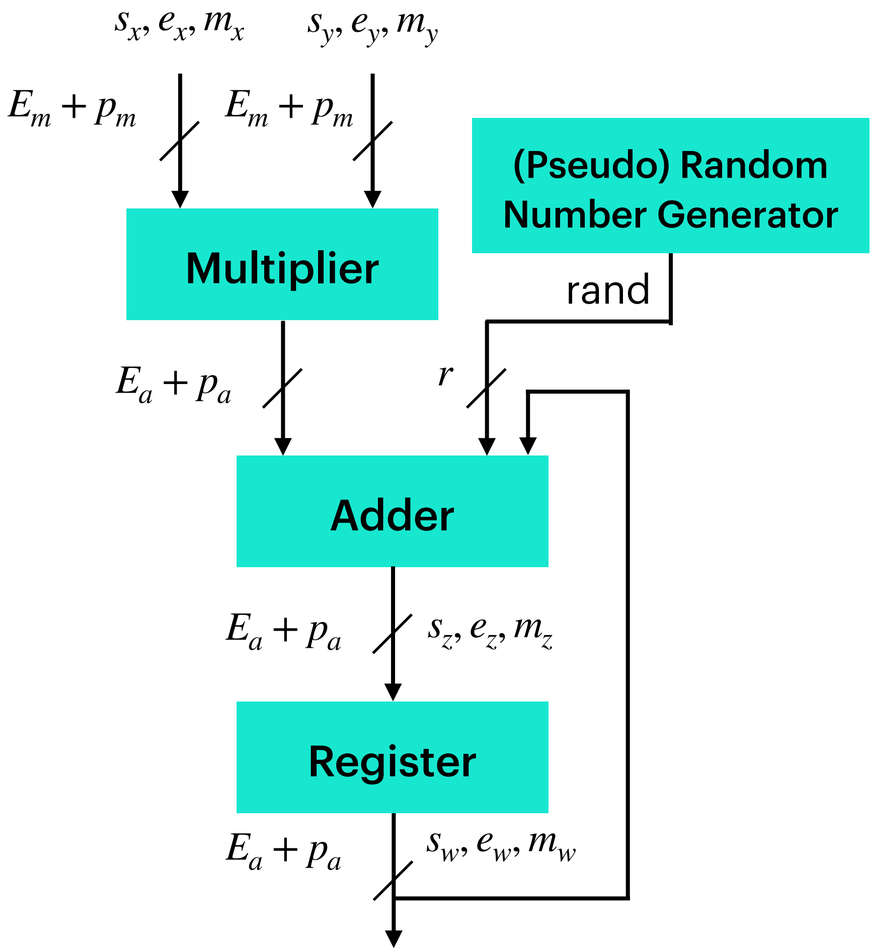


$$z = xy + z$$

Central to performing dot product and matrix multiplication

Classic MAC Unit Design with SR

Multiply-Accumulate (MAC) Unit Design



$$z = \mathrm{SR}\left(xy + w\right)$$

Central to performing dot product and matrix multiplication

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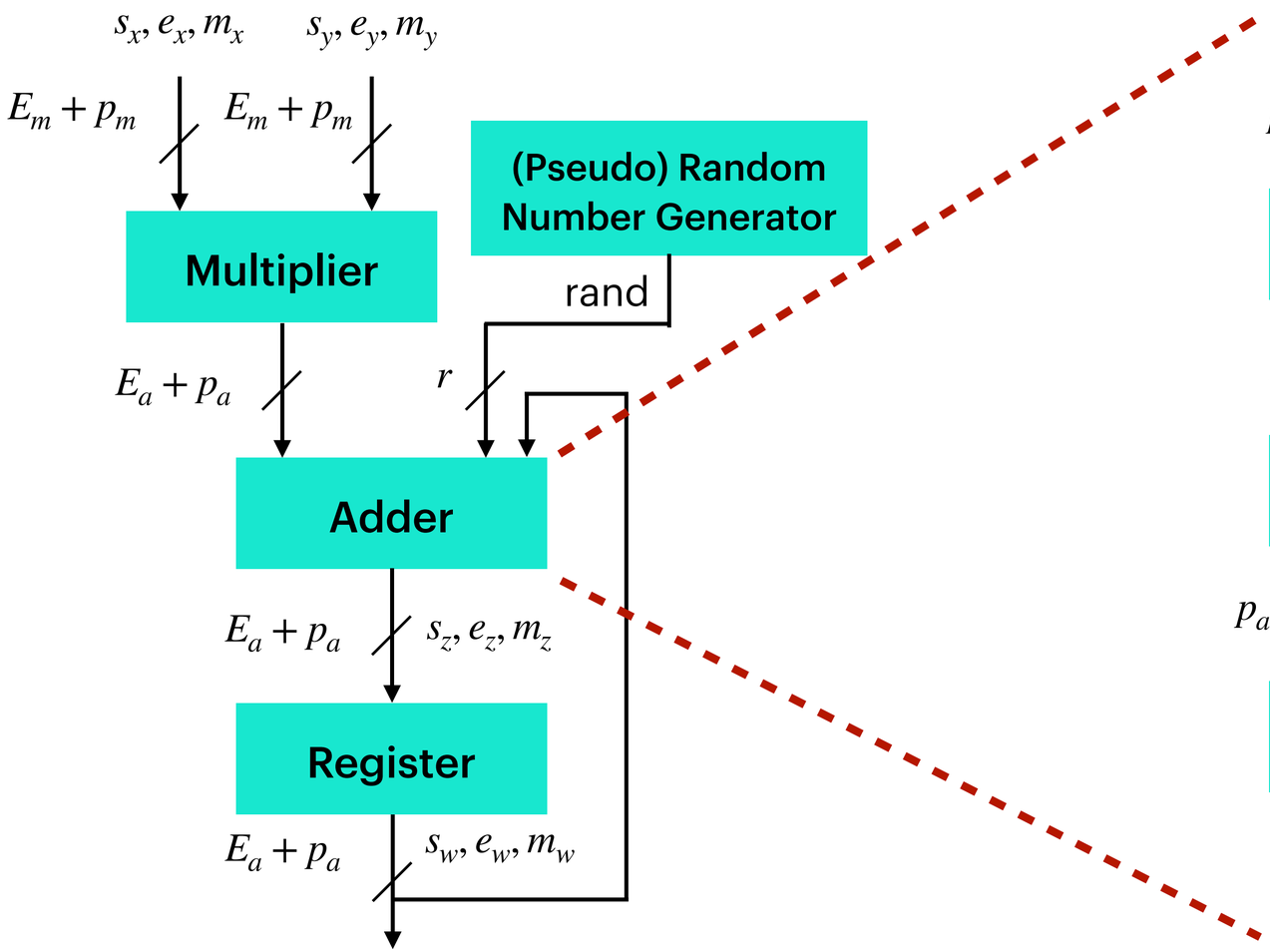
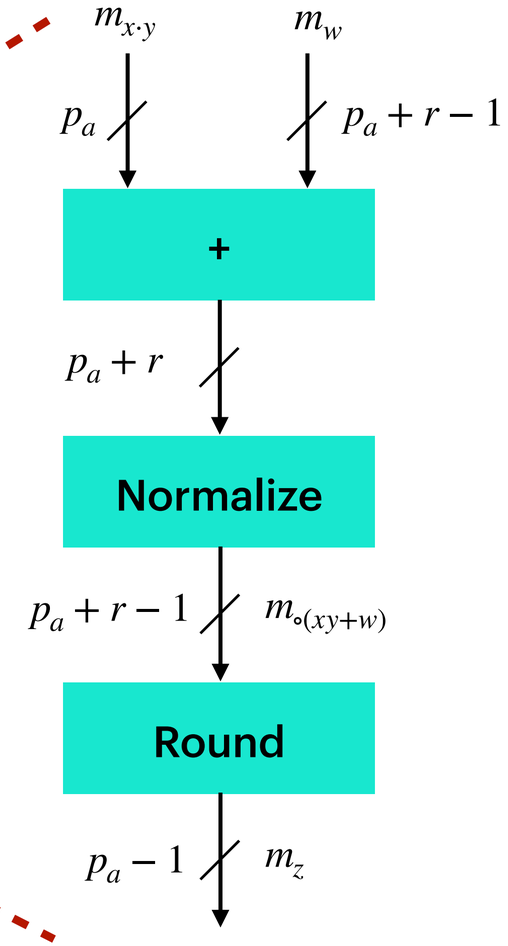


Adder Architecture



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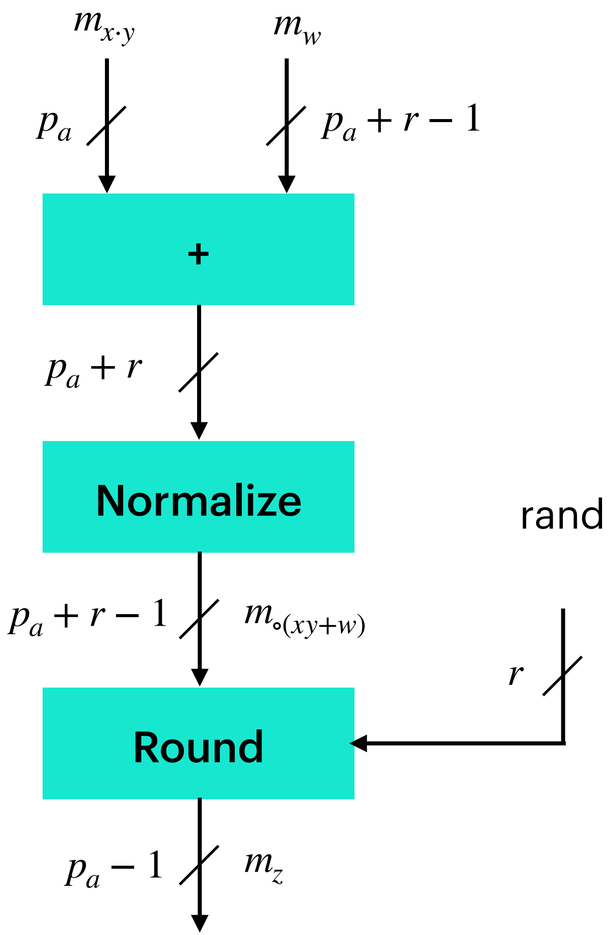
Central to performing dot product and matrix multiplication

Classic MAC Unit Design with SR

Classic SR Design



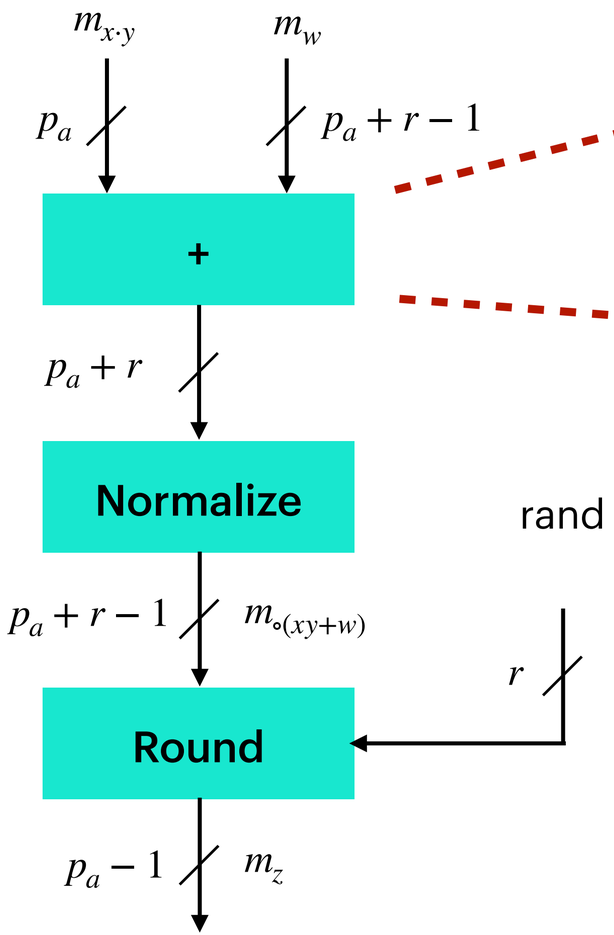
Toy Example: $a + b$

$$a = 1.111_{(2)} \cdot 2^8$$

$$b = 1.101_{(2)} \cdot 2^5$$

Classic MAC Unit Design with SR

Classic SR Design



Toy Example: $a + b$

$$a = 1.111_{(2)} \cdot 2^8 \qquad b = 1.101_{(2)} \cdot 2^5$$

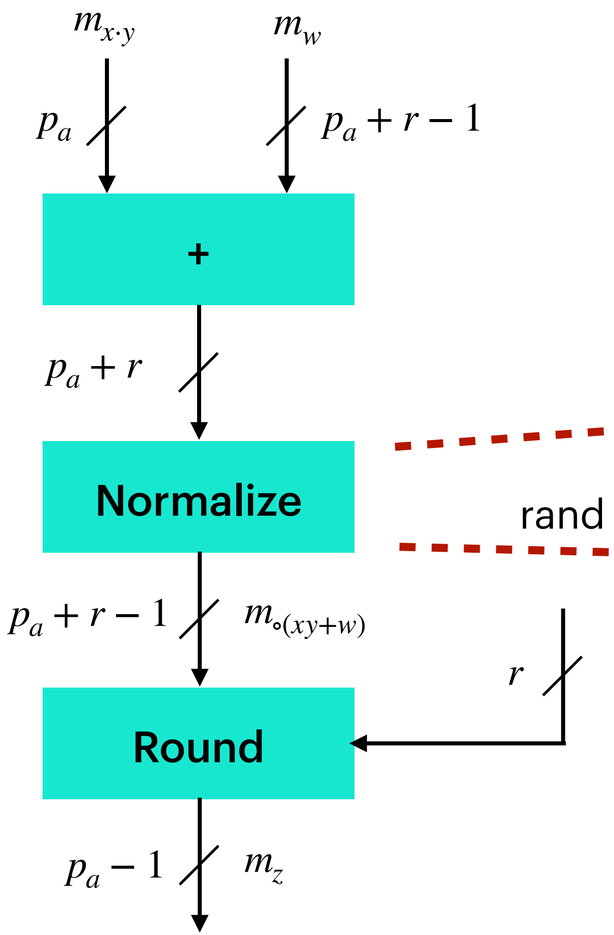
significand alignment →

$$a = 1.111000_{(2)} \cdot 2^8 \qquad b = 0.001101_{(2)} \cdot 2^8$$

significand addition $a + b = 10.000101_{(2)} \cdot 2^8$

Classic MAC Unit Design with SR

Classic SR Design



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$$a = 1.111_{(2)} \cdot 2^8 \quad \xrightarrow{\text{significand alignment}} \quad a = 1.111000_{(2)} \cdot 2^8$$

$$b = 1.101_{(2)} \cdot 2^5 \qquad\qquad b = 0.001101_{(2)} \cdot 2^8$$

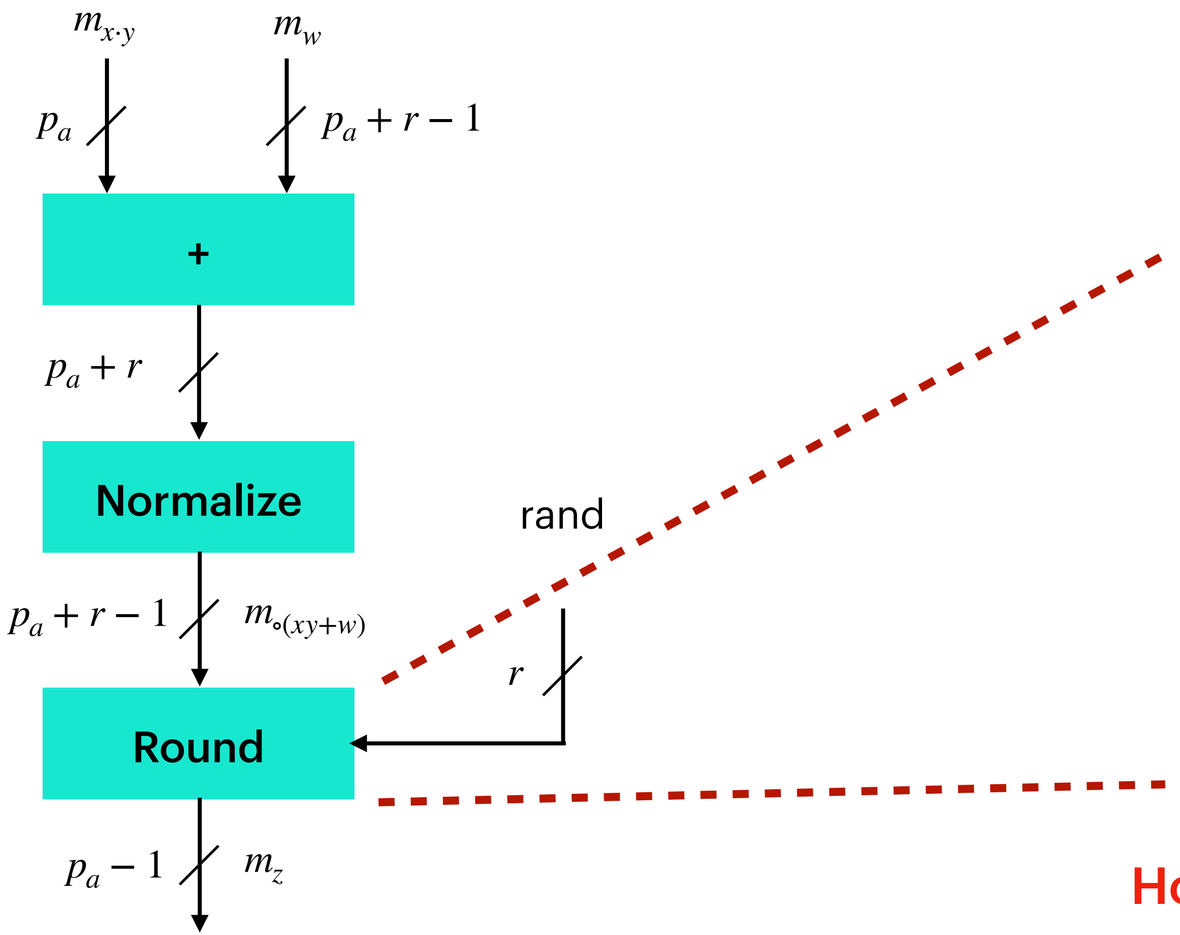
significand addition $\quad a + b = 10.000101_{(2)} \cdot 2^8$

normalization

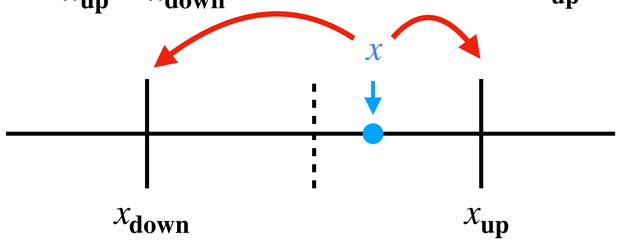
$$a + b = 1.0000101_{(2)} \cdot 2^9$$

Classic MAC Unit Design with SR

Classic SR Design



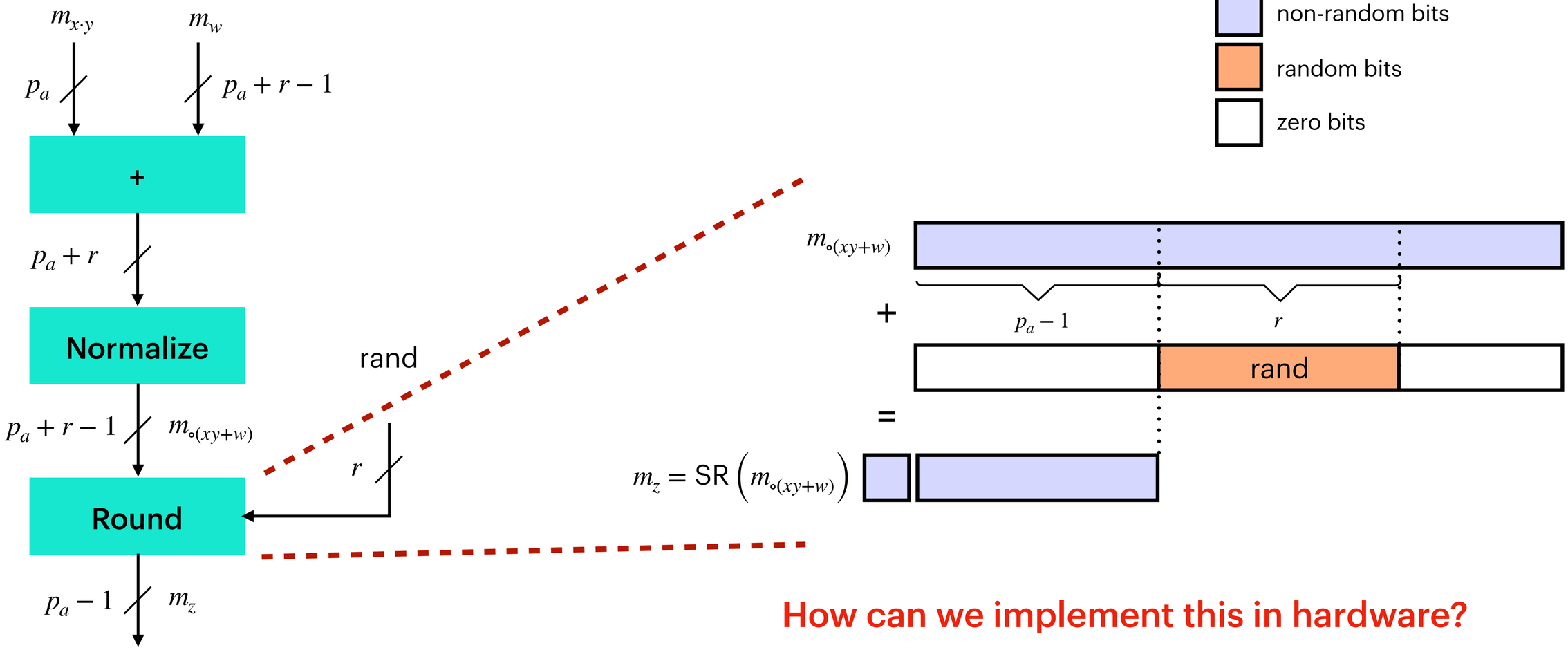
$$p_{\text{down}} = \frac{x_{\text{up}} - x}{x_{\text{up}} - x_{\text{down}}} \qquad p_{\text{up}} = \frac{x - x_{\text{down}}}{x_{\text{up}} - x_{\text{down}}}$$



How can we implement this in hardware?

Classic MAC Unit Design with SR

Classic SR Design



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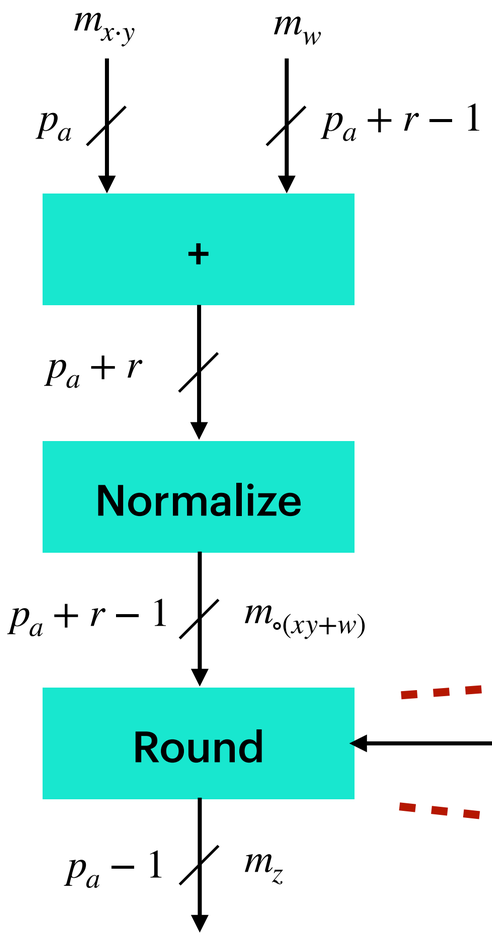

rand

r

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significand addition

$$a + b = 10.000101_{(2)} \cdot 2^8$$

normalization

$$a + b = 1.0000101_{(2)} \cdot 2^9$$

add random bits

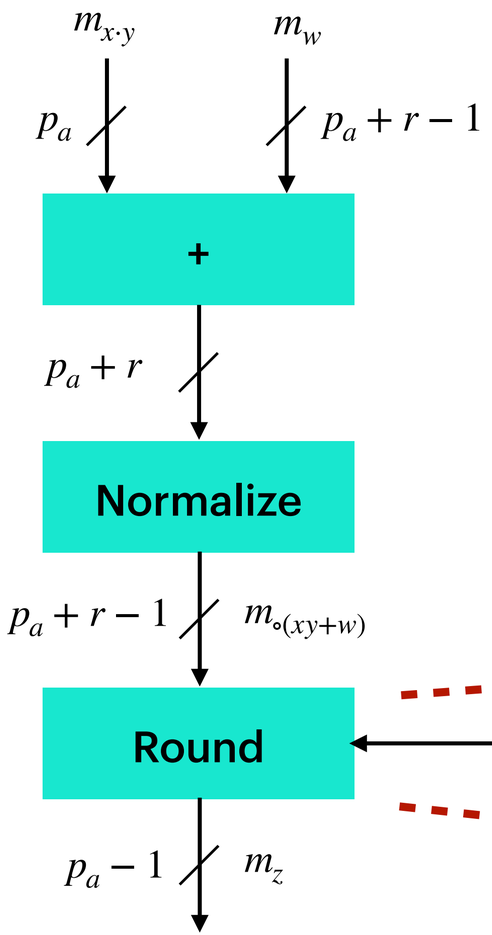
$$a + b = 1.0000101_{(2)} \cdot 2^9$$
$$\textbf{rand} = 0.0001101_{(2)} \cdot 2^9$$

round

$$\textbf{SR}(a + b) = 1.001_{(2)} \cdot 2^9$$

Classic MAC Unit Design with SR

Classic SR Design



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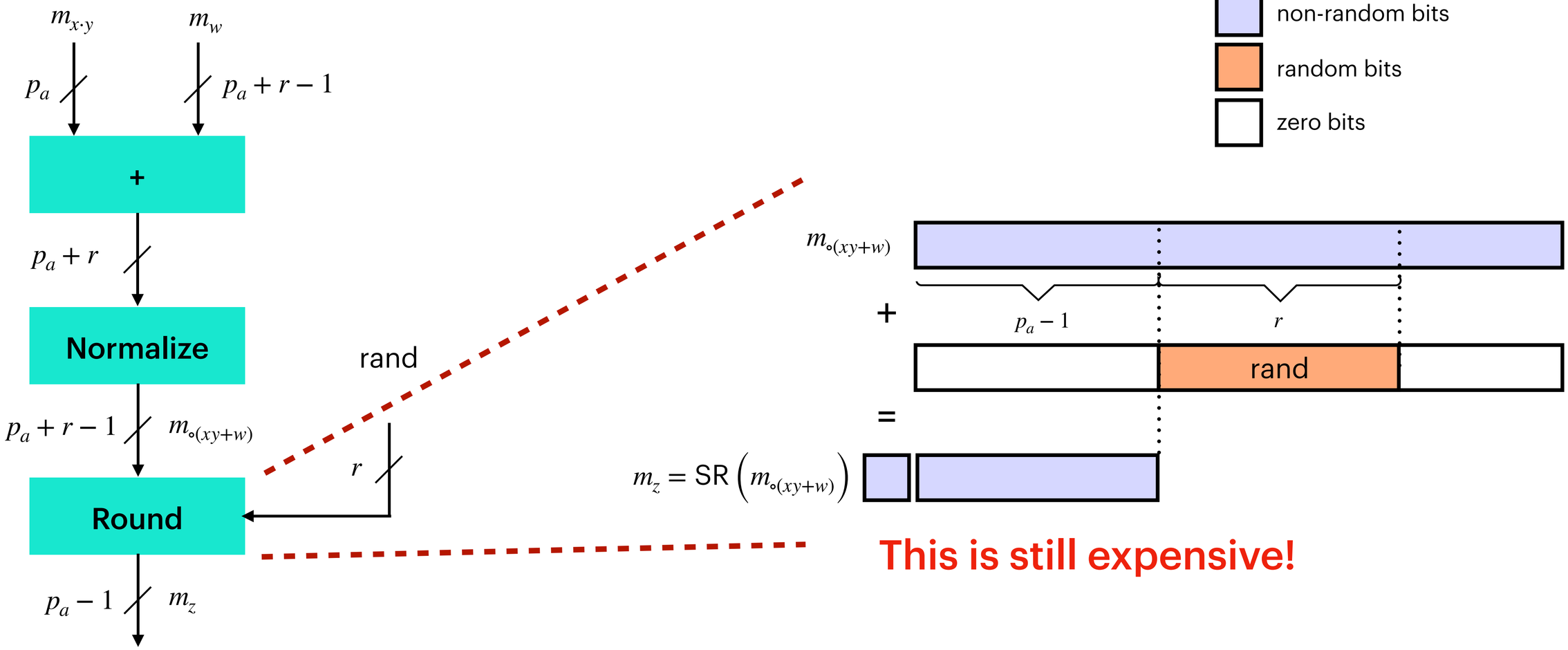
round

$$\textbf{SR}(a + b) = 1.000_{(2)} \cdot 2^9$$

Classic MAC Unit Design with SR

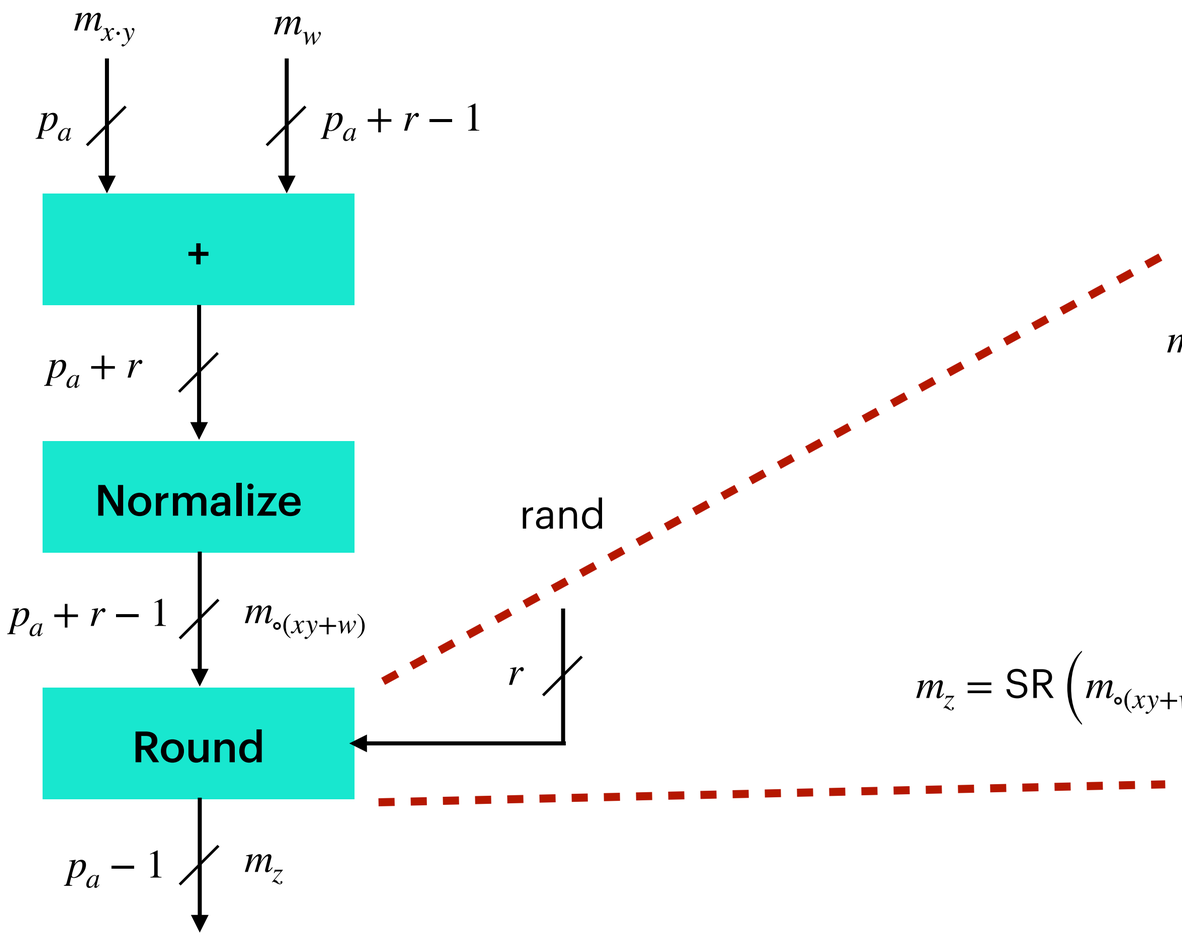
Classic SR Design

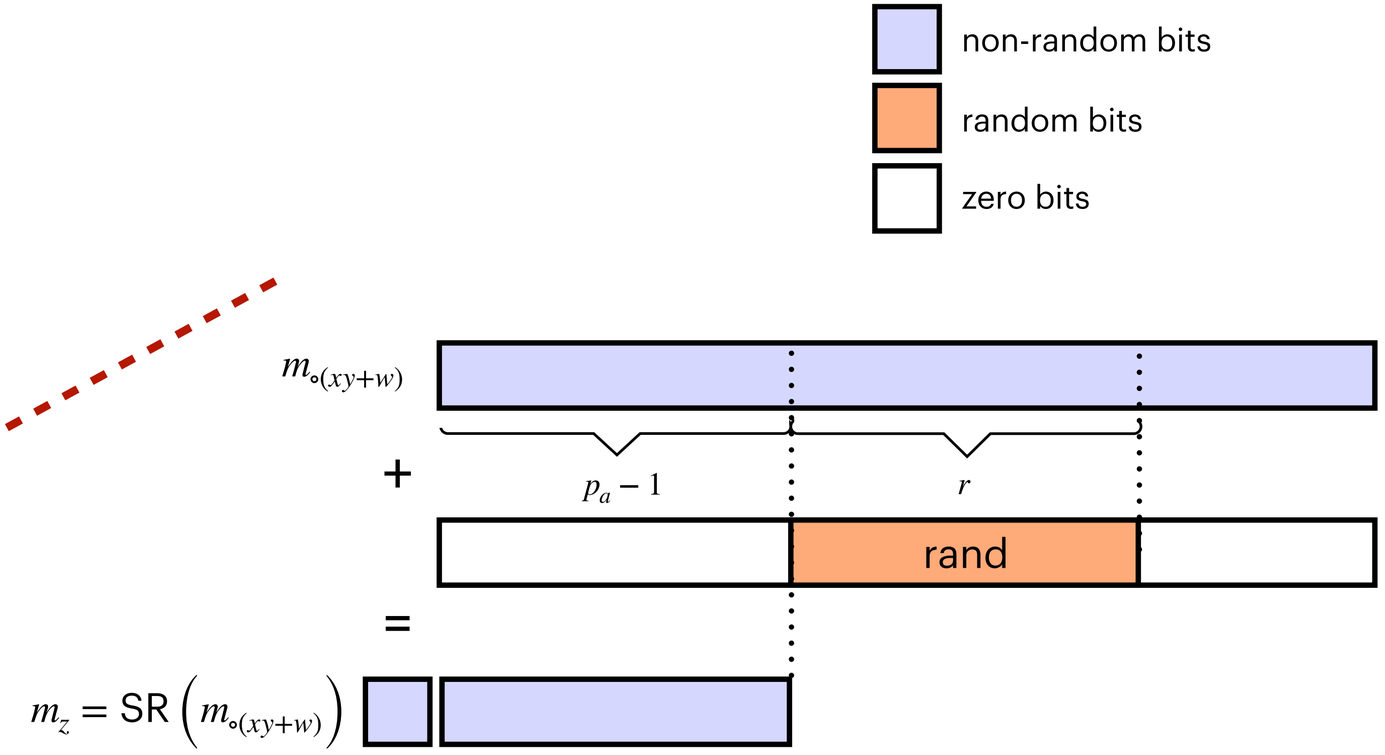


This is still expensive!

Classic MAC Unit Design with SR

Classic SR Design (lazy)





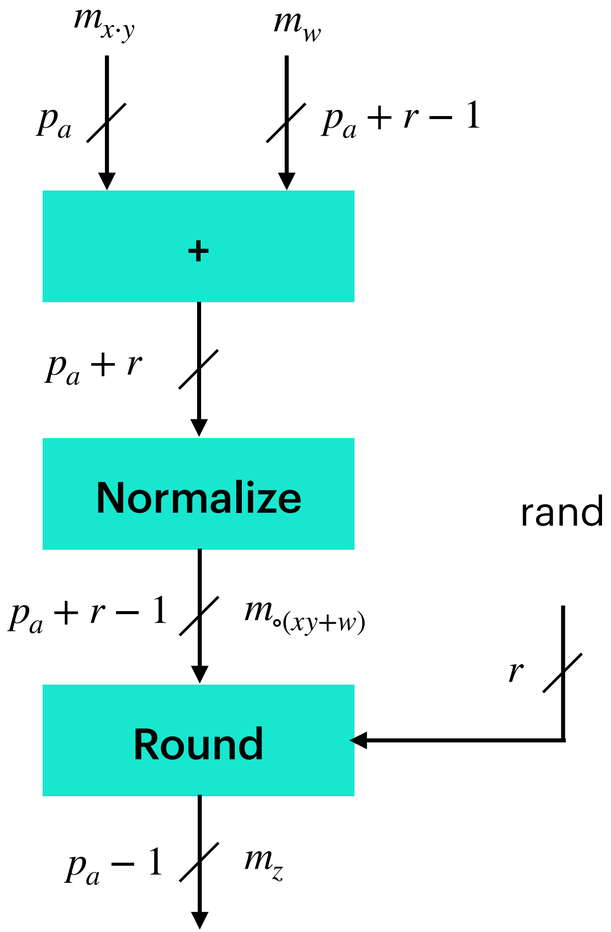
This is still expensive!

Idea: start rounding process early

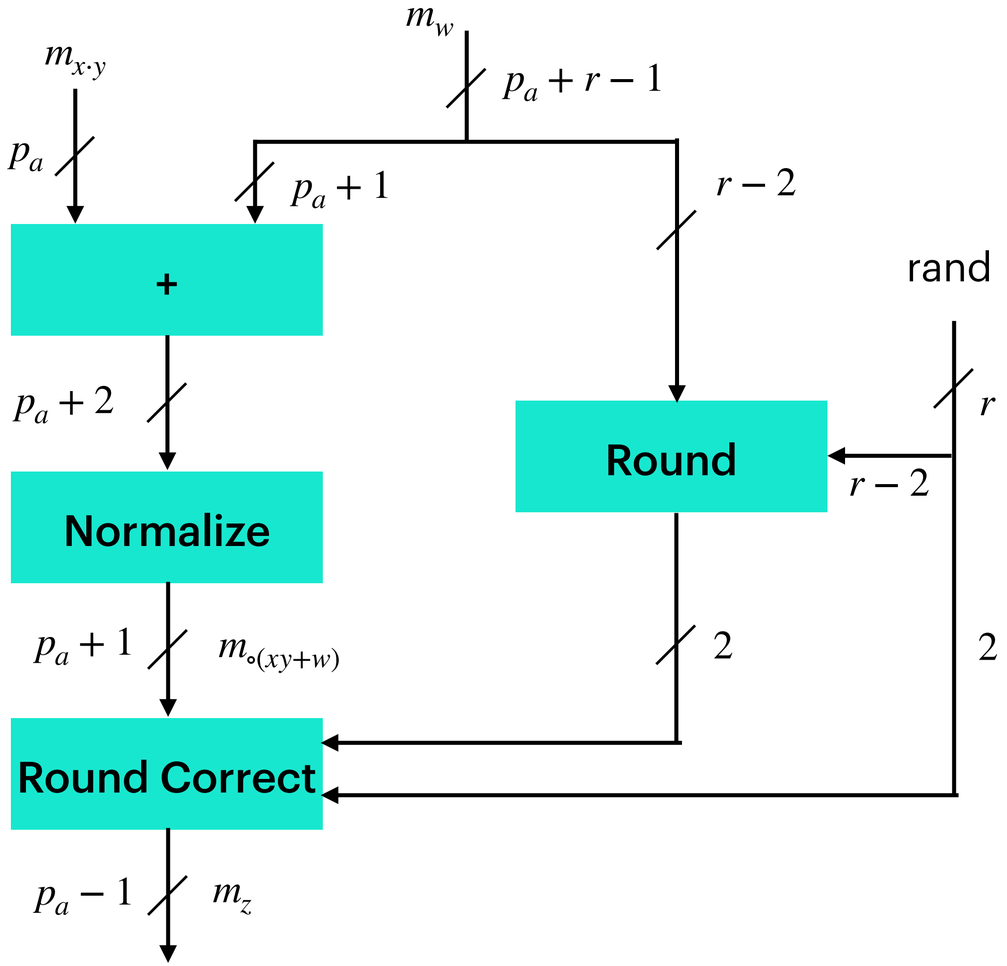
[1] A Stochastic Rounding-Enabled Low-Precision Floating-Point MAC for DNN Training, *S. Ben Ali, S.-I. Filip, O. Sentieys.*, in proceedings of the 2024 IEEE/ACM Design, Automation and Test in Europe (DATE) conference, Mar 2024

Eager MAC Unit Design with SR

Classic SR Design (lazy)



Proposed SR Design (eager)



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Eager MAC Unit Design with SR

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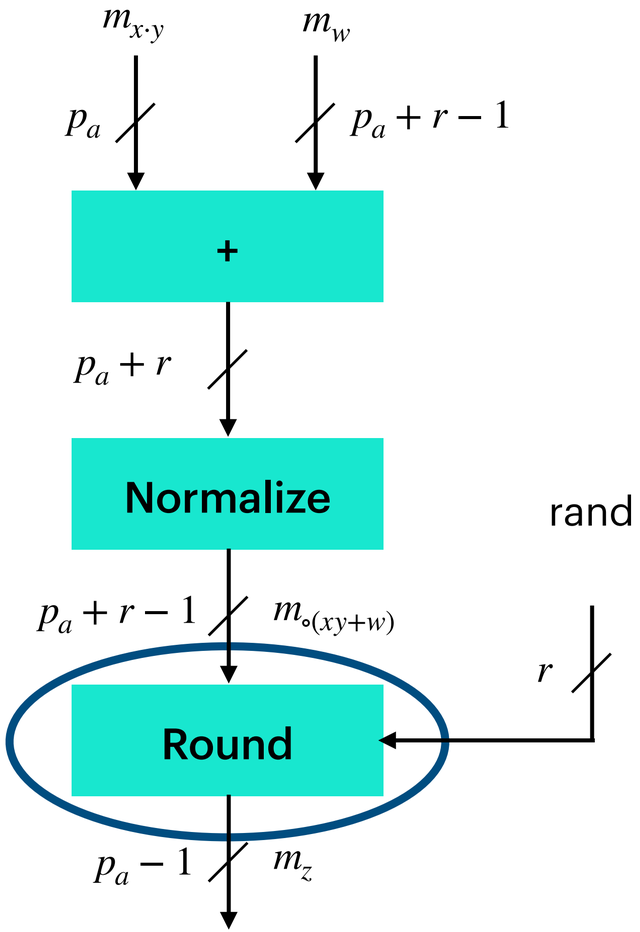
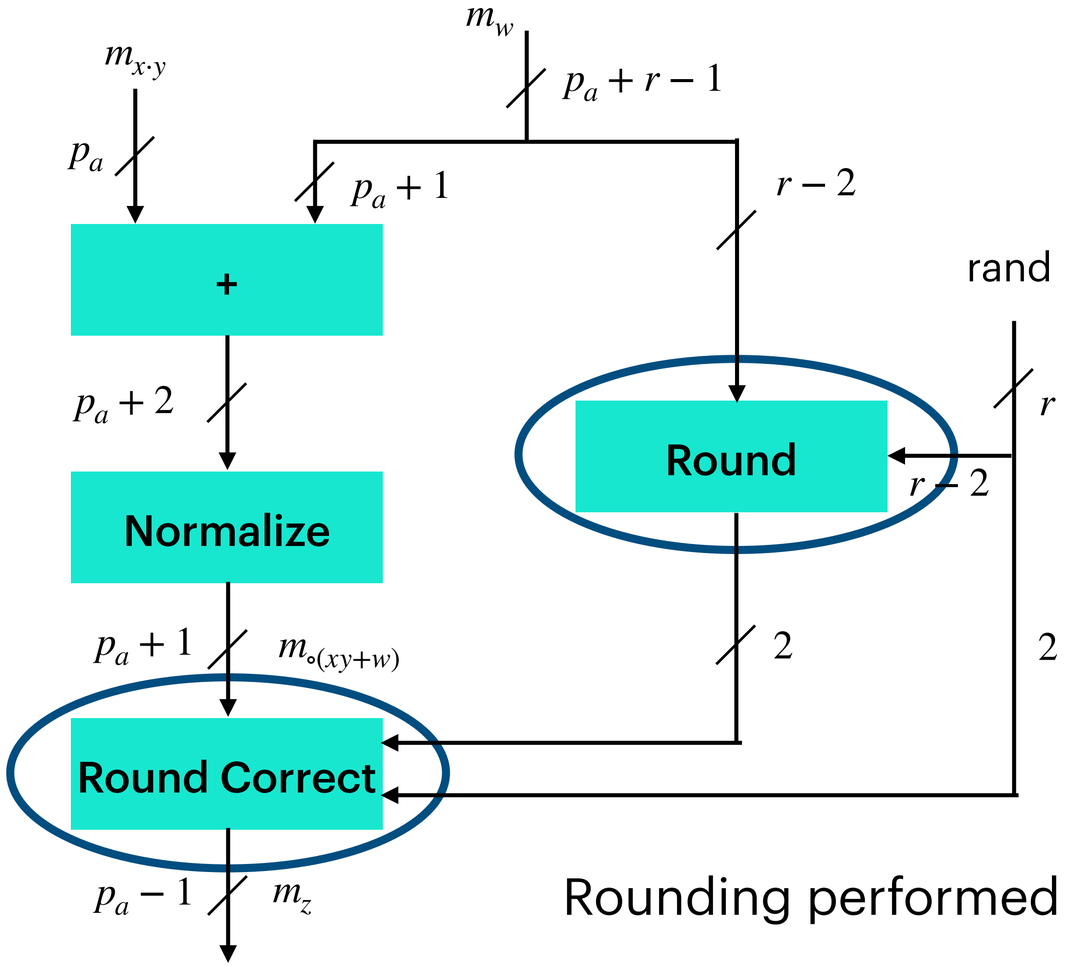


Proposed SR Design (eager)



Rounding performed in two stages

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Eager MAC Unit Design with SR

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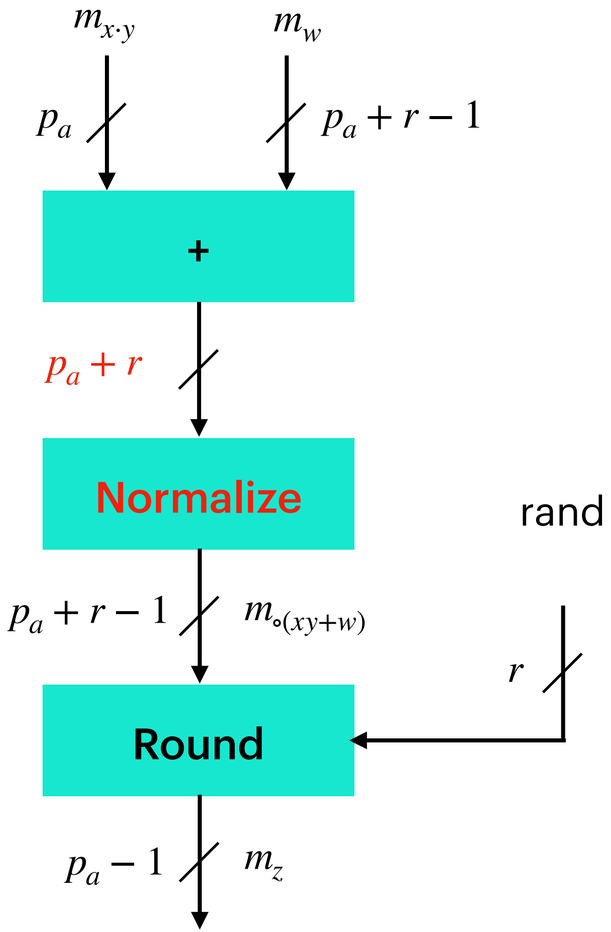
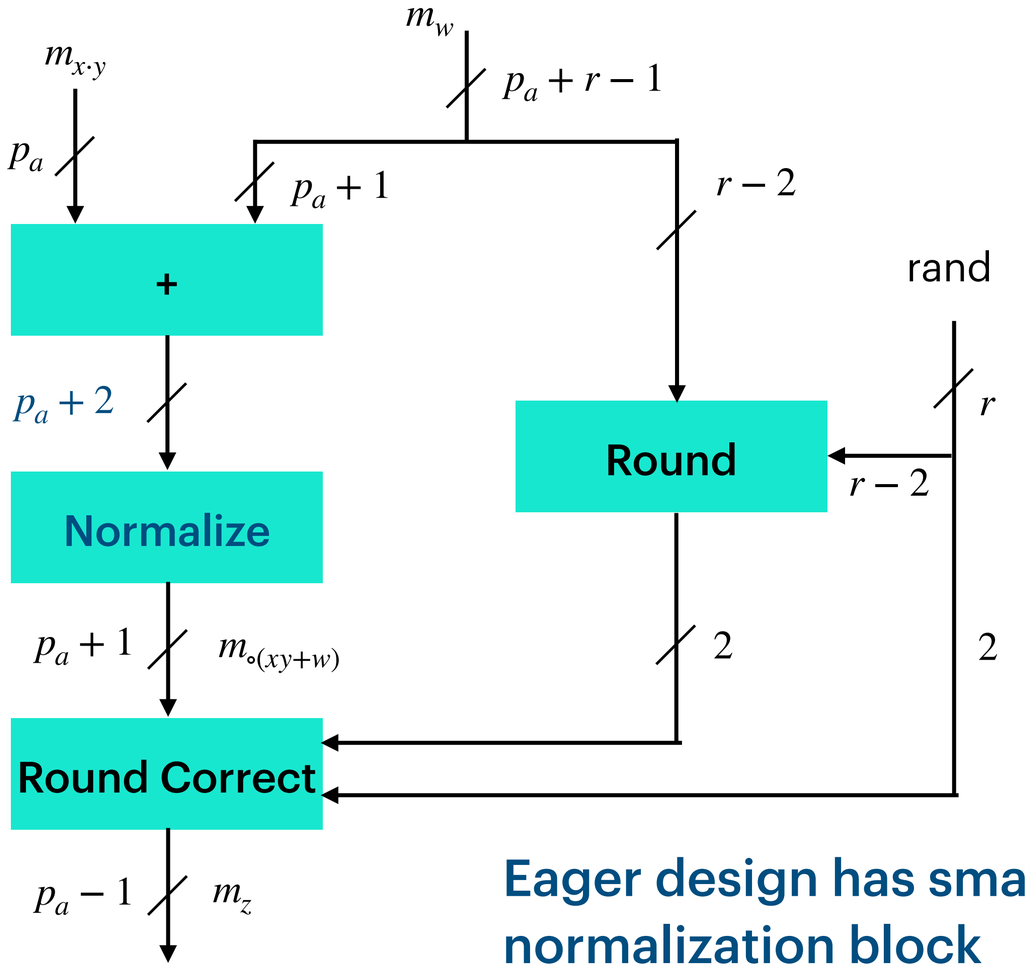


Proposed SR Design (eager)



Eager design has smaller normalization block

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Hardware Synthesis Results

FP12 (6,5) accumulator with SR

Synthesis results obtained using FDSOI 28nm technology

Achieve up to **18.5%** and **14.2%** savings in area and energy

➡larger savings for larger r (9 vs 18 here)

Can report up to **32.2%** delay reduction